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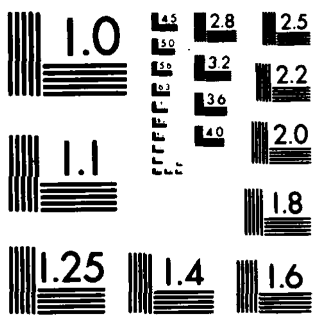

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83-0934	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) APPLICATIONS OF NON-SELF ADJOINT OPERATOR THEORY TO THE SINGULARITY EXPANSION METHOD (SEM) AND TO THE EIGENMODE EXPANSION METHOD (EEM) IN ACOUSTIC AND ELECTROMAGNETIC PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED Final Report: June 1, 1980 - May 31, 1983
		6. PERFORMING ORG. REPORT NUMBER Final Report: 80-0204
7. AUTHOR(s) Charles L. Dolph		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0204
9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Michigan Ann Arbor, Michigan 48109		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F; 2304/A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, DC 20332		12. REPORT DATE JULY 83
		13. NUMBER OF PAGES 97
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

(1) Singularity Expansion Method (SEM); (2) Eigenmode Expansion Method (EEM); (3) Scattering Theory; (4) Green's functions; (5) Exterior Dirichlet/Neumann Problem; (6) Spectral Properties of Operators

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Item 20 Continued

A conference summarizing ten years of work on the Singularity Expansion Method (SEM) and the Eigenmode Expansion Method (EEM) was held at the Carnahan House of the University of Kentucky in November 1980. The proceedings of this conference, originally due for publication in Fall 1981, did not appear until the Spring of 1982 in Volume 1, Number 4 in "Electromagnetics" (October-December 1981).

These proceedings contain two papers, one by C. L. Dolph and one by A. G. Ramm which reviews work on this subject. These papers are reproduced in Appendix I and II respectively of this report.

Subsequently C. L. Dolph submitted an expanded version of A. G. Ramm's paper to the Journal of Mathematical Analysis and Applications [Vol. 86, No. 2, April 1982]. This paper was entitled "Mathematical Foundations of the Singularity and Eigenmode Expansion Methods (SEM and EEM). In this author's opinion it represents the best available mathematical treatment of the subjects it covers. It is reproduced here as Appendix III.

Section 5 of this paper entitled Problems is still viable. There have been claims that, under some circumstances, the complex roots of the Green's function for the exterior Dirichlet or the Neumann Laplacian are simple. To this author's knowledge this is still an open problem.

AFOSR-TR- 83-0934

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AFOSR GRANT 80-0204

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Final Report

July 1983

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Department of Mathematics
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Final Scientific Report

A conference summarizing ten years of work on the Singularity Expansion Method (SEM) and the Eigenmode Expansion Method (EEM) was held at the Carnahan House of the University of Kentucky in November 1980. The proceedings of this conference, originally due for publication in Fall 1981, did not appear until the Spring of 1982 in Volume 1, Number 4 in "Electromagnetics" (October-December 1981).

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Section 5 of this paper entitled Problems is still viable. There have been claims that, under some circumstances, the complex roots of the Green's function for the exterior Dirichlet or the Neumann Laplacian are simple. To this author's knowledge this is still an open problem.

Although Ramm and Dolph have repeatedly emphasized that the Picard method is only valid for a restricted class of surfaces, all the work that this author has seen on equivalent circuits is based on the validity of the Picard method and consequently must be considered of limited value.

Two other topics have been treated since the above conference. The first of these appears in a paper "Converence of the T-matrix Approach to Scattering Theory" , Journal of Mathematical Physics Vol. 23 (6), June 1982, and its subsequent generalization in a preprint by the same title, coauthored by G. Kristanson, A. G. Ramm, and S. Ström. This represents work done at the Institute of Theoretical Physics in Goteborg, Sweden, in the summer of 1982. In contrast to the first paper on this subject, the second paper was not done under the auspices of this grant.

These papers are reproduced as Appendices IV and V, respectively.

One large area still under investigation involves the use of variational principles in these problems. It will perhaps be recalled that M. S. Agranovitch carefully avoids these even though he laid much of the foundations for the EEM and SEM in his appendix "Spectral Properties of the Diffraction Problems" which is contained in the book Generalized Methods of Normal Modes in Diffraction Theory by N. Voytovich, B. Katsenelenbaum and A. Sivov, Moscow (1977). More specifically he states "the formal Ritz method of finding stationary values of the functions (see Chapter III) was not analyzed" .

This problem has been around for a long time. A. G. Ramm, in "Variational Principles for Resonances, II" , Journal of Mathematical Physics Vol. 23 (6), 1982, developed an interesting approach similar to that used in his paper on the T-matrix. This paper is reproduced as Appendix VI .

This author has been exploring min-max theory in the hope of obtaining error bounds for scattering problems. Earlier papers

"A Saddle Point Characterization of the Schwinger Stationary Points in Exterior Scattering Problems", J. Soc. Indust. Appl. Math. Vol 5, No. 3, September 1957.

"The Schwinger Variational Principles for One-Dimensional Quantum Scattering" (with R. K. Ritt), Math. Zeitschrift Band 65 (1956), 309-326.

If the orientation of the saddle could be determined a priori, it appears possible to use relative cycle theory to obtain estimates. A search of the literature has not been helpful in that, while min-max theory is highly developed in the theory of games and in control theory, fuzzy set theory, and critical point theory, no error estimates appear to be known.

An earlier paper

"Symmetric Linear Transformations and Complex Quadratic Forms" (with J. E. McLaughlin and I. Marx), Comm. Pure Appl. Math. Vol. VII (1954), 621-632

still has not been generalized to the infinite case.

Two other earlier papers

"A Critique of Singularity Expansion and Eigenmode Expansion Methods" (with V. Komkov and R. A. Scott), Proc. of Conference on Acoustics, Electromagnetic and Elastic Wave Scattering: Focus on T-Matrix Approach, Pergamon Press (1980), 453-462.

"On the Relationship Between the Singularity Expansion Method the Mathematical Theory of Scattering" (with S. K. Cho), IEEE Trans. on Antennas and Propagation, V. AP-28, No. 6 (1980), 888-897.

are still pertinent to any evaluation of the SEM and EEM methods.

APPENDICES

This document contains six papers

(singularity expansion method)
(eigenmode expansion method)

APPENDIX I

ON SOME MATHEMATICAL ASPECTS OF SEM, EEM AND SCATTERING

C. L. Dolph, *The University of Michigan, Department of Mathematics, Ann Arbor, MI 48109*

ABSTRACT

The relationship between the integral equations usually used in SEM and the scattering matrix is examined. Alternate integral equations which exhibit only the poles of the S matrix are given. Examples are used for illustration for a solvable case.

The analytic Fredholm theorem in Banach spaces is discussed and its advantages for numerical calculations emphasized.

The relationship between EEM, SEM and the theory of nonselfadjoint operators is briefly discussed.

INTRODUCTION

The ideas lying behind the Eigenmode Expansion Method (EEM) appear to have been introduced for the first time by Kacenelenbaum in 1969 [6.4]. The Singularity Expansion Method (SEM) was first introduced by Baum in 1971 [3.1] and shortly thereafter independently he introduced EEM [3.4]. The best review paper of these USSR contributions is that due to Voitovic, Kacenelenbaum and Sivov [6.11] and that of the USA's contributions (in this author's opinion) is that of Baum [2.2]. The most complete review of the Russian work through 1976 is the Russian book [6.11] by the above three Russian authors. This book also contains a mathematics appendix by M.S. Agranovic.

A glance at the official bibliography makes it clear that extensive work has been undertaken and completed since these beginnings, and more will be discussed in these preceeding.

In view of the extensive publications the author thought it might be most useful to provide a brief guide to some of the recent mathematical developments without excessive detail and without proofs.

This work was partially supported by the Air Force Grant 800204

Electromagnetics 1:375-383, 1981
0272-6343/81/040375-09$2.25

SEM

For the scalar wave equation an exterior Dirichlet problem would be formally given by the first four of the following equations. The fifth equation is the solution given in terms of generalized eigenfunctions which are distorted plane waves playing the role of the plane waves used in the Fourier integrals which occur when no obstacle is present. The functions $\alpha(k)$ and $\beta(k)$ are related to the initial conditions. This last formula has been rigorously established by Shenk [6.92] in a manner similar to that used by Ikebe [6.35] for the quantum mechanical case. Explicitly the generalized eigenfunctions are defined by (6), (7) and (8). As will be discussed below several different methods are available for the construction of $V_{\pm}$

(1) $$\Delta U = \frac{\partial^2 u}{\partial t^2}$$

(2) $$U(x,0) = f_1$$

(3) $$\frac{\partial u}{\partial t}(\underline{x},0) = f_2$$

(4) $$U = 0 \text{ on } \Gamma$$

(5) $$U(\underline{x},t) = \frac{1}{(2\pi)^{3/2}} \int \phi_{\pm}(\underline{x},k)\,[\alpha(k)\, e^{ikt} + \beta(k)\, e^{-ikt}]\, d^3k$$

(6) $$(\Delta + k^2)\phi_{\pm} = 0$$

(7) $$\phi_{\pm} = 0 \text{ on } \Gamma$$

(8) $$\phi_{\pm} = e^{ik\underline{x}\cdot\underline{\xi}} + V_{\pm}(\underline{x},k)$$

In contrast to the operator theory approach employing the continuous spectrum SEM employs the Laplace transform which, after a suitable rotation in the s-plane, can be defined by (9), (10), (11), and (12). Condition (10) is one form of the radiation condition which is need to guarantee uniqueness of all k, $\mathrm{Im}\, k \geq 0$.

(9) $$V(x,k) \triangleq \int_0^{\infty} U(x,t)\, e^{ikt}\, dt$$

(10) $$\Delta V + k^2 V = -f$$

(11) $$V = 0 \text{ on } \Gamma$$

(12) $$\frac{\partial V}{\partial |x|} - ikV = O(|x|^{-1}) .$$

The function $V(x,k)$ is sought in terms of the Green's function as given in equation (13). The Green's function in this equation is not the well-known Green's function of free space but is determined by (14), (15), (16), and its domains of analytic and meromorphicity are given by the next two statments (17) and (18). See Dolph, McLeod and Thoe (6.25).

(13) $$V(x,k) = \int_\Omega G(x,y,k)f(y)dy$$

where G satisfies

(14) $$(\Delta + k^2)G = -\delta(x-y) \quad \text{in } \Omega$$

(15) $$G = 0 \quad \text{on } \Gamma$$

(16) $$\frac{\partial G}{\partial |x|} - ikG = O(|x|^{-1})$$

(17) $$G(x,y,k) \quad \text{analytic} \quad \text{Im } k \geq 0$$

(18) $$G(x,y,k) \quad \text{meromorphic} \quad \text{Im } k < 0$$

Once $V(y,k)$ has been found the solution of the original problem can be given in terms of the inverse Laplace transform.

If $a > 0.5$, $\text{Im } k > -b$, $b > 0$ and

(19) $$|V| \leq \frac{c}{1+|k|^a}, \quad |\text{Re } k| \to \infty .$$

Then for $0 < \gamma < b$

$$U(x,t) = \frac{1}{2\pi} \int_{-i\gamma-\infty}^{-i\gamma+\infty} V(x,k)e^{-ikt}\, dk$$

Pushing the contour down yields [6.69]

(20) $$U(x,t) = \sum_{j=1}^{n} e^{-ik_j t} V(x,k_j) + O(e^{-|I_m k_n| t})$$

The function V and k's which occur in this asymptotic formula are the complex eigenfunctions and eigenvalues:

(21) $$\Delta V_j + k_j^2 V_j = 0 \quad \text{in } \Omega$$

(22) $$V_j = 0 \quad \text{on } \Gamma$$

V_j grows exponentially in x .

Several comments are now in order:

(i) The estimate (19) is valid for the Dirichlet problem if the body is (a) star-shaped and (b) non-trapping in the sense of Lax and Phillips (6.52).

(ii) The method is not very useful since it involves the construction of the Green's function and then the determination of its poles.

(iii) It is an open problem to find conditions when the asymptotic series (20) will actually converge.

Instead in SEM it is usual to employ the methods of potential theory. For the exterior time-independent Dirichlet problem corresponding to the time dependent problem we have been considering up until now, this involves consideration of the following set of equations which employ the known Free space Green's function.

$$(\Delta + k^2)V = 0$$

$$V = -V_{inc} \quad \text{on} \quad \Gamma$$

$$\phi_0(x,y) = \frac{1}{2\pi} \frac{e^{ik|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|}$$

$$(23) \quad V(\underline{x},k) = \int_\Gamma \frac{\partial}{\partial n_y} \phi_0(\underline{x},\underline{y}) d(\underline{y}) d\sigma_y$$

$$(24) \quad I + B(k) \stackrel{\Delta}{=} d(\underline{y}) + \lambda \int_\Gamma \frac{\partial}{\partial n_y} \phi_0(\underline{x},\underline{y}) d(\underline{y}) d_y = 0$$

$$\lambda = \lambda(k) = -1$$

Using the Fredholm alternative the poles are sought as non-trivial solutions of the homogeneous integral equation (24). Those which may occur for real k correspond to eigenvalues of the associated interior Neumann problem. As such, as we shall see, they occur because of the double-layer assumption and can be eliminated by other assumptions. As shown by Dolph and Wilcox, see Dolph [6.96] they do not contribute to the scattered field nor do they appear in it for any separable case.

The homogeneous integral equation which occurs here can be treated mathematically several different ways. Marin [3.9] employed Carleman's Hilbert space theory but the analytic Fredholm theorem attributed to Steinberg [6.94] is perhaps the most convenient since it is applicable in more general Banach spaces. Since matrix approximations are used in the numerical calculation of the poles the choice of the Banach space of continuous functions is perhaps the most convenient. See Dolph and Cho [6.22] for a fuller discussion. For a Hilbert space the proof of the analytic Fredholm theorem can be found in Reed and Simon [6.84].

Analytic Fredholm Theorem - Steinberg [6.94]

Let $O(B)$ = set of bounded operators on the Banach space and let K be an open connected subset of the complex plane. $T(K)$ is analytic in K if for each $k_0 \in K$

$$T(k) = \sum_0^\infty T_n (k-k_0)^n \qquad T_n \in O(B)$$

Theorem. If $T(k)$ is an analytic family of compact operators for $k \in K$, then either $I - T(k)$ is nowhere invertible in Ω, or else $[I-T(k)]^{-1}$ is meromorphic in K.

If B is a separable Hilbert space, the residues are finite rank operators.

One way of eliminating the poles which are not intrinsic to the exterior scattering problem is to replace the Ansatz (23) of the double layer by the complex combination of a double and single layer as used by Brakhage and Werner for the Dirichlet problem [6.11] and by Kussmaul [6.50] for the Neumann problem. In the latter case additional difficulties need to be overcome because of the high order of the singularity.

For the Dirichlet Problem the Ansatz

$$V_+(\underline{x},\underline{k}) = \int_\Gamma \left(\frac{\partial}{\partial n_y} - i\tau\right) \phi_0(\underline{x},\underline{y})\mu(\underline{y})d_y\sigma$$

leads to the homogeneous integral equation:

$$\text{(25)} \qquad \mu(x) + \int_\Gamma \left(\frac{\partial}{\partial n_y} - i\tau\right) \phi_0(\underline{x},\underline{y})\mu(\underline{y})d_y\sigma = 0$$

$$\tau(x) = 1 \quad \text{for} \quad \text{Re } k \geq 0$$

$$= 0 \quad \text{for} \quad \text{Re } k < 0 .$$

This equation has only trivial solution for $\text{Im } k \geq 0$ and hence the only non-trivial solutions can occur for $\text{Im } k < 0$ and as Ramm [6.72] has shown these occur at the poles of a Green's function and are in fact the intrinsic poles of SEM.

The non-trivial solutions of this last equation for $\text{Im } k < 0$ also agree with the poles of the S matrix. The S matrix is generally thought to contain all intrinsic properties and in fact Lax and Phillips have given two proofs of the fact that the S matrix uniquely determines the obstacle for the Dirichlet problem -- see Theorem (5.6) cr. [6.52], Chapter V.

For the problem here it can be shown that the $V_-(x,h)$ of (8) and the S matrix are related by the formulas: [In the last equation the integral operator is compact].

$$X = r\underline{\theta},\ \underline{\xi} = k\underline{w}$$

$$V_-(r\theta,kw) = \frac{e^{-ikr}}{r}\,[s_-(\theta,k,w) + O(1)]$$

$$S(k)m(\ ) = m(\theta) + \frac{ik}{2\pi}\int_{|w|=1} m(w)\overline{s_-(\theta,k,w)^*}dSw$$

The complex eigenvalues are poles of $S(k)$.

Derivations of these formulas can be found in Lax and Phillips [6.52], in Schmidt [6.87] for the quantum case of the Schrodinger equation and for a very general case in Shenk and Thoe [6.9]. A physical derivation of the last formula is due to Saxon [6.86] is also contained in Dolph and Cho [6.22]. This last paper also contains an appendix in which a heuristic derivation of the mathematical theory of scattering initiated by Jauch is given.

For the cylinder (24) becomes

$$\mu(a,\phi_0) + \frac{ika}{4}\int_0^{2\pi} \frac{\partial}{\partial(ka)} H_0^{(1)}\left(ka \sin \frac{\theta_0}{2}\right)\ (a,\mu_0+\phi_0)d\theta_0 = 0$$

and has solutions given by

$$\mu(a,\phi_0) = \sum_{-\infty}^{\infty} \frac{2(-1)^{n-1}J_n(ka)e^{in\phi_0}}{J_n^1(ka)H_n^{(1)}(ka)}$$

The complex roots of the Hankel function are intrinsic, those of the derivative of the Bessel function well-known to be those of the associated interior Neumann problem.

The Brakhage-Werner equation corresponding to (25) is

$$\sigma(a,\phi_0) - \frac{ika}{4}\int_0^{2\pi}\left[\frac{\partial}{\partial(ka)} - i\eta\right] H_0^{(1)} \quad ka\,\frac{\sin\theta_0}{2}\quad \sigma(a,\theta_0+\phi_0) = 0$$

with a solution exhibiting only intrinsic poles; namely

$$\sigma(a,\phi_0) = \frac{2}{\pi ka}\sum_{-\infty}^{\infty}\frac{(-1)^{n-1}J_n(ka)e^{in\phi_0}}{H_n^{(1)}(ka)\,[J_n^1(ka) - iJ_n(ka)]} .$$

For this problem the complex eigenfunctions are

$$V(r,\theta) = e^{im\theta}\,H_m^{(1)}(k_0 r)$$

where

$$(\Delta + k_0^2)V = 0, \quad V - 0,\ r = a$$

and the scattering matrix is given, as shown by Shenk and Thoe [6.91] to be

$$S(k)\left\{\sum_{-\infty}^{\infty} a_m e^{im\theta}\right\} = -\sum_{-\infty}^{\infty}\frac{H_m^{(2)}(ka)a_m e^{im\theta}}{H_m^{(1)}(ka)} .$$

In most cases it is necessary to resort to matrix approximation or to have methods for the calculations of the poles. In the case of the former, Ramm [6.72] has established the following:

> Poles coincide with k_j for which $I + B(k)$ of (24) is not invertible. Let $\{f_j\}$ be an orthonormal basis in $H = L^2(\Gamma)$. Then if
>
> $$\mu_n \triangleq \sum_1^n c_j f_j$$
>
> $$b_{ij} \triangleq \langle [I+B(k)]f_i, f_j\rangle$$
>
> It follows that
>
> $$\sum_{j=1}^n b_{ij}(k)c_j = 0$$
>
> Let $k_m^{(n)}$, $m = 1,2,3...$, be the roots of
>
> $$\det b_{ij}(k) = 0$$
>
> Then the limits $\lim_{n\to\infty} k_m^{(n)} = k_m$ exist and are the poles of the poles of the Greens function G . Every pole of G can be obtain in this way.

EEM

For this same time-independent Dirichlet problem the Eigenmode Expansion method would involve the following:

$$\Delta\phi + k^2 = 0$$

$$\phi = g \quad \text{on} \quad \Gamma$$

$$A(k)\phi = \int_\Gamma \frac{e^{ik|x-y|}}{4\pi|x-y|}\phi(y)dy$$

<u>Ansatz</u>

$$A(k)\phi = g \quad \text{on} \quad \Gamma$$

$$A(k)\phi_n = \lambda_n(k)\phi_n \quad \text{on} \quad \Gamma$$

<u>Picard</u> method gives

$$\phi = \sum_1^\infty \frac{\langle g,\phi_n \rangle}{\lambda_n}\phi_n$$

when is this valid?

The Picard process is certainly valid for the cylinder and the sphere. In fact as first noted by Kacenelenbaum, Sivov and Voitovic [6.11] for the cylinder they are explicitly given in the case of even θ by

$$\phi_n(\theta) = \cos n\theta$$

$$\lambda_n(k) = \frac{i\pi a}{2} H_n(ka)J_n(ka)$$

$$\phi_n(r, \ ) = H_n^{(1)}(ka)J_n(kr)\cos n\theta \ , \ r \leq a$$

$$H_n^{(1)}(kr)\ J_n(ka)\ \cos n\theta \ , \ r \geq a$$

While Dolph [6.20] appears to have been the first to suggest the use c non-self-adjoint operators in scattering problem Agranovic in [6.1], [6.2], [6.3] and Ramm [6.72], [6.73] appear to be the first to systematically apply this idea. Ramm in particular considered the Hilbert space case. That is Let

$$H = L_2(\Gamma)$$

$$\langle f,g \rangle = \int_\Gamma f(\underline{x})\overline{g(\underline{x})}\ d$$

Then $\langle Af,g \rangle = \langle f,\overline{Ag} \rangle$. This is real symmetry and $A \neq A^*$ i.e., A is non-self adjoint.

<u>Question</u>. When is the Ansatz correct? Sufficient condition: $AA^* - A^*A = 0$...(1) i.e., A is <u>normal</u>. Then an orthogonal basis can be found in $H = L_2(\Gamma)$. Here (1) requires

$$\int_\Gamma \frac{\sin k(|\underline{x}-\underline{t}|-|\underline{t}-\underline{y}|)}{|\underline{x}-\underline{t}|\ |\underline{t}-\underline{y}|}\,d\sigma_{\underline{t}} = 0$$

This last condition can be shown to be satisfied by the cylinder and sphere but not for the ellipse or ellipsoid. In particular then any EEM theory results which use the Picard process and are used to construct equivalent circuits are suspect in general.

Before entering into what is known in the case when the operator A is not normal the relation between SEM and EEM when the Picard process is valid should be mentioned.

In [2.2] Baum discussed the matrix case and showed that every zero of $\lambda(k)$ was a pole. More generally:

The Relation between SEM and EEM

Theorem (Ramm). The poles of $G(x,y,k)$ are zeros of the eigenvalues

$$\lambda_n(k) = 0 \qquad G_0 = \frac{1}{4\pi}\,\frac{e^{ik|x-y|}}{|x-y|} \tag{26}$$

$$G(x,y,k) = G_0(x-y) - \int_\Gamma G_0(x,s,k)\,\mu(s,y,k)\,d\sigma_s \tag{27}$$

where $\mu = \dfrac{\partial G}{\partial n_s}$

$$B\mu = \frac{\partial}{\partial n_x}\,\frac{e^{ik|x-y|}}{2\pi|x-y|}\;\mu(y)\,d\sigma_y$$

$$\mu + B\mu = 2\,\frac{\partial G_0}{\partial n}$$

If the operator is not normal the situation is much more complicated in general. One usually has to contend with root vectors as they occur in the Jordan normal form. The simplest example of their occurrence is in the matrix solution of ordinary differential equations with repeated roots. The questions of when are the root vectors complete, when do they form a basis are difficult in general. There is one case when there is a simple theorem concerning completeness, namely if the operator is dissipative. An operator A is said to be dissipative if

$$\operatorname{Im}\langle A\phi,\phi\rangle \geq 0 .$$

Many of the operators in mathematical physics are dissipative. For example the free space Green's function is:

$$A\phi = \int \frac{e^{ih|x-y|}}{4\pi|x-y|}\;\phi(y)\,dy .$$

One has for real k

$$\operatorname{Im}\langle A\phi,\phi\rangle = \iint \frac{\sin k|x-y|}{4\pi|x-y|}\,\phi(x)\,\bar{\phi}(y)\,d^3x\,d^3y$$

and the delta like behavior of the kernel implies that

$$\operatorname{Im}\langle A\phi,\phi\rangle \cong \int |\phi(x)|^2 d^3x \geq 0 .$$

A rigorous proof of this can be found in Dolph (6.20).

Ramm (6.72) has established a completeness theorem for such operators which are compact and nuclear.

Before stating his result note that if $s_n(A)$ are the eigenvalues of $(A^*A)^{1/2}$ a compact operator is called nuclear if $\sum_1^\infty s_N(A) < \infty$.

Theorem 1. If $A = P + N$ where P is positive and compact, and N is dissipative and nuclear. Then the root vectors of A are complete.

A simple pertinent example is given by

$$A\phi = \frac{1}{4\pi}\int \frac{e^{ik|x-y|}}{|x-y|}\,\phi(y)d\sigma_y \quad .$$

by taking $P\phi = \frac{1}{4\pi}\int \frac{\phi(y)d\sigma y}{|x-y|}$ and $N\phi = (A-P)\phi$.

More information on root vectors and basic can be found in the reference (6.34).

Finally, space limitations do not permit detailed discussion of many topics important to the further development of this subject. These include the weak perturbation of compact operators see (6.42, (6.55), (6.75) as well as variational principles (6.74) and papers in press by Ramm.

APPENDIX II

ON THE SINGULARITY AND EIGENMODE EXPANSION METHODS (SEM AND EEM)

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See JMAA 86, #2, 1982 85-62

INTRODUCTION

This is a brief summary of the invited talk given by the author at the Lexington (November 1980) meeting. The purpose of this paper is to formulate the mathematical problems important for the SEM and EEM, to answer several basic questions and to draw attention to certain unsolved problems. Some new results are also reported. The detailed presentation of the talk was sent to the Mathematical Notes (ed. C. E. Baum) and submitted for publication in the J. Math. Anal. Appl. The bibliography is not complete: only the papers in which the results mentioned in this article appeared were included in the bibliography.

1. STATEMENT OF THE EEM AND SEM

Let Ω be an exterior domain with a smooth closed boundary Γ, D be the corresponding interior domain, $G_0 = \frac{\exp(ik|x-y|)}{4\pi|x-y|}$, $r = |x|$, $Ag = \int_\Gamma G_0(s,s',k)g(s')ds'$ and $u = \int_\Gamma G_0(x,s,k)g(s)dx$. The function u solves the problem

$$(\nabla^2+k^2)u = 0 \text{ in } \Omega,\ u|_\Gamma = f,\ r(\partial u/\partial r-iku) \to 0 \text{ as } r \to \infty \tag{1}$$

provided that

$$Ag = f \tag{2}$$

If one uses the Laplace transform variable p, then $p = -ik$, and the half plane $\mathrm{Re}p > 0$ corresponds to the half-plane $\mathrm{Im}k > 0$. Engineers [6.41] - [2.2] tried to solve (2) by the formula $g = \sum_{j=1}^{\infty} \lambda_j^{-1}c_jf_j$, where, $Af_j = \lambda_jf_j$, $|\lambda_1| \geq |\lambda_2| \geq \dots$ and $f = \sum_{j=1}^{\infty}c_jf_j$. This can be done if $A = A^*$ is selfadjoint on $H = L^2(\Gamma)$. The operator A in (2) is nonselfadjoint. Therefore: 1) it may have no eigenvectors (e.g. $Ag = \int_0^x g dx$ on $H = L^2[0,1]$),

Supported by AFOSR 800204.

Electromagnetics 1:385-393, 1981
0272-6343/81/040385-09$2.25

2) it may have not only eigenvectors but also root vectors [6.69], [6.68],
3) it is an open question whether one can expand an arbitrary function $f \in H$ in the series of eigenvectors and root vectors of A. Of course one is interested in the rate of convergence of the series in eigen and root vectors and in algorithms for calculation of the root vectors and eigenvalues of A. The outlined method (EEM) has the following merits: 1) instead of problem (1) with a continuous spectrum in the unbounded domain we consider problem (2) with a discrete spectrum on the compact manifold Γ, 2) the resonance properties can be conveniently studied by the EEM. A mathematical study of the EEM was originated in [6.72], [6.73], [6.71].

In order to describe SEM consider the problem

$$u_{tt} = \nabla^2 u \text{ in } \Omega, \; u|_\Gamma = 0, \; u|_{t=0} = 0, \; u_t|_{t=0} = f \tag{3}$$

The solution of this problem takes the form

$$u = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ikt)v(x,k)dk, \; v = \int_\Omega G(x,y,k)f\, dy \tag{4}$$

where G is the Green function for problem (1),

$$G = G_0 - \int_\Gamma G_0(x,s,k)\mu ds, \; \mu = \frac{\partial G(s,y,k)}{\partial n_s} \tag{5}$$

$$[I+T(k)]\mu = 2\frac{\partial G_0}{\partial n_s}, \; T(k)\mu = 2\int_\Gamma \frac{\partial G_0}{\partial n_s} \mu\, ds' \tag{6}$$

We assume that $f \in C_0^\infty(\Omega)$. From (5), (6) it follows that G is finite-meromorphic in k. This means that G is meromorphic on the whole complex plane k and its Laurent coefficients are degenerate kernels (finite rank operators on H). If $\Omega \subset R^3$ then G is analytic in Im $k > 0$. Thus v is meromorphic in k and analytic in Im $k \geq 0$. Let us assume that

$$|v| \leq c(b)(1+|k|)^{-a}, \; a > 1/2, \; |\mathrm{Re}\, k| \to \infty, \; \mathrm{Im}\, k = b \tag{7}$$

where b is an arbitrary const;

$$|\mathrm{Im}\, k_j| \to \infty \text{ as } j \to \infty, \; |\mathrm{Im}\, k_1| \leq |\mathrm{Im}\, k_2| \leq \ldots \tag{8}$$

where k_j are the poles of v.

Note that (7)$\Rightarrow$(8). From (7), (8) it follows (by moving the contour of integration in (4) down) that

$$u(x,t) = \sum_{j=1}^{N} c_j(x,t)\exp(-ik_j t) + o(\exp(-|\mathrm{Im}\, k_N|t)), \; t \to +\infty \tag{9}$$

Here $c_j(x,t)\exp(-ik_j t) = \mathrm{Res}\, v(x,k)\exp(-ik_j t)$ at $k = k_j$; $c_j(x,t) = O(t^{m_j - 1})$, where m_j is the order of the pole k_j. Thus

we see that (7), (8) and the meromorphic character of v are sufficient for the SEM of the form (9) (asymptotic SEM). It is an open question if the series

$$u(x,t) = \sum_{j=1}^{\infty} c_j(x,t) \exp(-ik_j t) \qquad (10)$$

converges. The validity of the EEM was discussed in [6.69], [6.68].

2. COMPLEX POLES OF GREEN'S FUNCTIONS

We saw in Section 1 that complex poles k_j are important. It is interesting to answer the following questions: 1) how does one calculate the poles? 2) are the poles simple? 3) do the poles depend continuously on the scatterer? 4) Can one identify the scatterer from the knowledge of complex poles? 5) what can be said about location of the poles and asymptotic behavior of the large poles nearest to the real axis? 6) are there any monotonicity or other features in the behavior of the purely imaginary poles? 7) What are the properties of the resonant states (natural modes corresponding to the complex poles)? 8) What is the relationship between the poles and the eigenvalues used in the EEM?

We give some answers to the above questions. Three different methods for calculation of the complex poles were given in [6.71-2], [6.68] and [6.74]. The first method is most general. It reduces the problem to calculation of the values k_j at which a certain operator of the type $I + T(k)$, where $T(k)$ is a compact analytic operator function, is not invertible. These k_j can be found by a projection method. The method is described in [6.71-2] (see also [6.68]). Its convergence is proved [6.71] The second method is a variational principle for complex poles: k_j^2 are the stationary values of the functional

$$K(u) = \langle \nabla u, \nabla u\rangle / \langle u,u\rangle, \text{ where } \langle u,v\rangle = \lim_{\varepsilon \to +0} \int \exp(-\varepsilon|x|\ln|x|)uv\,dx \qquad (10')$$

and the integral is taken over Ω. In [6.74] a certain system of test functions was suggested but the rigorous justification of the numerical procedure given in [6.74] is an open problem. In [6.68] a variational principle for the spectrum of compact nonselfadjoint operators was given. In [6.71] it was proved that the complex poles of the Green's functions are the complex zeros of the eigenvalues of certain integral operators. This gives the third method of calculation of the poles: first, one calculates the eigenvalues, then one looks for their zeros. No numerical results are known for the third method. It would be interesting to make numerical experiments and to compare all the three methods.

It is an open question whether the poles are simple. In [6.71] it was proved that the poles are simple if the surface is of such shape that the operator A in (2) is normal, i.e. $AA^* = A^*A$. In [6.73] it was proved that this is so if Γ is a sphere or a straight line (linear antenna). Recently the author gave a simple example of a multiple pole in the problem with third boundary condition:

if $(\nabla^2+k^2)u = 0$ in $r = |x| \geq 1$, $\partial u/\partial r - 2u = \cos\theta$ on $|x| = 1$, $r(\frac{\partial u}{\partial r} - iku) \to 0$ as $r \to \infty$, then $k = -2i$ is a pole of order 2 of $u(x,k)$. Generically multiple poles are exceptions because small perturbations of the shape of the scatterer can destroy multiple poles. On the other hand, since the poles depend continuously on Γ (see [6.68] for precise definitions and proofs) it seems possible that by continuous variation of Γ one can make a multiple pole out of 2 simple poles by merging. Nevertheless, no proof is known that the Green's function of the exterior Dirichlet Laplacian has multiple poles for some Γ.

We have already mentioned that the poles depend continuously on Γ. It is not known whether the set of complex poles determines the scatterer uniquely. A discussion of this question is in [6.76] and [*]. Some information on location of the poles is available: in [6.70] it was proved that the domain $\{\text{Im } k < 0, |\text{Im } k| < a\log|\text{Re } k| + b, a > 0\}$ is free from the complex poles of the Green's function of the Schrödinger operator with a compactly supported potential; in [6.54] a similar result was proved for the poles of the Green's function of the exterior Dirichlet Laplacian; in [6.5] some heuristic arguments are given to show that the domain $\{\text{Im } k < 0, |\text{Im } k| < a|\text{Re } k|^{1/3} + b, a > 0\}$ is free from the poles of the Green's function of the exterior Dirichlet and Neumann Laplacians provided that Γ is strictly convex and smooth; if Γ is not smooth (say, Γ is a polygon) then there exists a series of poles k_j such that $|\text{Im } k_j| = O(\log|\text{Re } k_j|)$ as $j \to \infty$ [6.6].

In [6.53] it was proved that there exist infinitely many purely imaginary poles of the Green's functions of the exterior Dirichlet or Neumann Laplacian and

$$cR_1^2 \leq \liminf_{y\to\infty} y^{-2} N(y) \leq \limsup_{y\to\infty} y^{-2} N(y) \leq cR_2^2$$

where $c = 1.138370\ldots\ldots$, $N(y)$ is the number of purely imaginary poles with $|\text{Im } k_j| < y$, the obstacle is star-shaped (this means that all points of Γ can be seen from a point in D) and R_1, R_2 are the radii of spheres inscribed in and circumscribing D, respectively. It is pointed out in [*] that if $D_2 = qD_1$, $q > 1$ then $y_j^{(1)} = qy_j^{(2)}$, where $-iy_j^{(1)}$ $(-iy_j^{(2)})$ are the poles of the Green's function of the exterior Dirichlet Laplacian in $\Omega_1(\Omega_2)$, $\Omega_j = R^3\setminus D_j$, $j = 1, 2$, where $R^3\setminus D$ denotes the complement to D in R^3. Therefore in this case $N_2(y) \geq N_1(y)$ and $y_1^{(1)} > y_1^{(2)}$, where $y_1^{(j)}$ are the moduli of the purely imaginary poles with minimal moduli. In [6.53] Theorem 3.5 on p. 751 says that $N_2(y) \leq N_1(y)$. This statement contradicts: 1) the above argument, and 2) the case when D_1 and D_2 are concentric balls and one can calculate $N_1(y)$ and $N_2(y)$ for $y >> 1$ and verify that

$N_2(y) > N_1(y)$. The argument in [6.53] can be used if the assumption $0 \subset 0_s$ is replaced by the assumption $0 \supset 0_s$. We mention this because in the literature one can find references and citations of Theorem 3.5 from [6.53] in its wrong form. Using arguments from [6.53] and assuming that D_j, $j=1, 2$ are star-shaped and that $D_1 \subset D_2 \subset D_3$, one can see that $N_1(y) \leq N_2(y) \leq N_3(y)$. Here we used the corrected version of Theorem 3.5 from [6.53]: if $D_1 \subset D_2$ and D_1 is star-shaped, then $N_1(y) \leq N_2(y)$. This theorem is actually proved in [6.53] so that the misstatement of Theorem 3.5 in [6.53] is just a misprint.

Concerning the behavior of the resonant states, that is the solutions of the homogeneous problem (1) for $k = a - iy$, $y > 0$, $f(x) = 0$), satisfying the asymptotic condition

$$u = |x|^{-1} \exp(ik|x|)\sum_{j=0}^{\infty} |x|^{-j} f_j, f_j = f_j(n,y),\ n = x \cdot |x|^{-1}, \quad (11)$$

at infinity, one can prove the following proposition: if $u \exp(-y|x|)|x| \to 0$ as $|x| \to \infty$ then $u \equiv 0$. From this it follows that the resonant states (scattering modes) corresponding to a complex pole $k = a - iy$ grow at infinity exactly as $O(\exp(y|x|)|x|^{-1}$. See also [6.43] Theorem 3. The relationship between SEM and EEM is given in the following proposition ([6.71-2], [6.68]): the set of the complex poles of the Green's function of the exterior Dirichlet Laplacian coincide with the set of complex zeros of the eigenvalues $\lambda_n(k)$ of the operator A defined in (2). It is not known at this time whether the order of a pole can be calculated from the multiplicity of zeros. One can construct other operators with the eigenvalues vanishing at the complex poles [6.68].

3. "ORTHOGONALITY" OF THE EIGENMODES AND RESONANT STATES

By eigenmodes (EM) we mean the eigenfunctions of the operator A defined in (2). This is a nonselfadjoint operator on $H = L^2(\Gamma)$ with the property $[Af,g] = [f,Ag]$, where $[f,g] = (f,\bar{g}) = \int_\Gamma fg\, ds$, $(\cdot,\cdot)$ is the inner product in $L^2(\Gamma)$, the bar denotes complex conjugation. Suppose that $Af_j = \lambda_j f_j$, $[f_j,f_j] \neq 0$, $j = 1, 2, \ldots$ and the set $\{f_j\}$ forms a basis of H. Then any $f \in H$ can be represented as $f = \sum_{j=1}^{\infty} c_j f_j$ and $c_j = [f,f_j]$. This can be proved exactly as in the case of orthogonal Fourier series if one takes into account that $[f_j,f_m] = 0$ for $j \neq m$. The last formula follows from the identity $0 = [Af_j,f_m] - [f_j,Af_m] = (\lambda_j-\lambda_m)[f_j,f_m]$ if $\lambda_j \neq \lambda_m$. If $\lambda_j = \lambda_m$ one can choose f_j, f_m so that $[f_j,f_m] = 0$ for $j \neq m$. Thus the coefficients in the EEM can be easily calculated. If the root vectors are present the formulas

for the coefficients in root vectors can also be calculated explicitly [*].

"Orthogonality" of the resonant states corresponding to different complex poles k_1, k_2 holds in the following sense: $\langle u(x,k_1), u(x,k_2)\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is defined in (10') (See [6.74] and [*] for details).

4. NONSMOOTH BOUNDARIES

The usual proof of the meromorphic nature of the Green's function of the exterior Laplacian requires smoothness of Γ. Indeed, it is based on the integral equation (6) and on the theorem about the meromorphic nature of the operator $(I+T(k))^{-1}$ [6.80-3]. If this theorem it is assumed that $T(k)$ is a compact operator function analytic in k. If the surface Γ has edges or conical points, the operator $T(k)$ in (6) is no longer compact. Nevertheless the theory is still valid provided that there are no cusps on Γ. This follows from the proposition (see [*] for details): if $T(k) = T + Q(k)$ is an operator function on a Hilbert space H, where $Q(k)$ is analytic in k for $k\in\Delta$, where Δ is a connected open set in the complex plane, $|T|_{ess} < 1$ and $I + T(k)$ is invertible at some point, then $(I+T(k))^{-1}$ is finite meromorphic in Δ, (finite meromorphic means that the Laurent coefficients are operators of finite rank). By $|T|_{ess}$ we mean inf $\|T-K\|$, where K runs through the set of all compact operators on H.

It is known [6.12] that $|T(0)|_{ess} < 1$ provided that there are no cusps on Γ. One can now apply the above proposition and conclude that μ in (6) (and therefore G; see (5)) is meromorphic and its Laurent coefficients are degenerate kernels.

5. EXAMPLES, COMMENTS

1. A symmetric (with respect to the form $[f,g]$ defined in section 3) nonselfadjoint operator can have root vectors. Example: $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $[x,y] = x_1y_1 + x_2y_2$. $(A-\lambda I)^{-1}$ has a pole of order 2 at $\lambda = 0$. The corresponding eigenvector is $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and the root vector is $\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$.

2. The fact that the algebraic problem to which an original integral equation was reduced (e.g. by a projection method) has eigenvalues does not guarantee that the original equation has. (See [6.68], [6.69] and [*] for details and sufficient conditions under which the eigenvalues of the algebraic problem converge to the eigenvalues of the original problem.)

3. The operator $(I+T(k))^{-1}$ can have multiple poles and be diagonalizable (i.e. $T(k)$ has no root vectors).

Example: $T(k) = \begin{pmatrix} -1 + k^2 & 0 \\ 0 & k^2 \end{pmatrix}$, $(I+T(k))^{-1} = \begin{pmatrix} k^{-2} & 0 \\ 0 & (1+k^2)^{-1} \end{pmatrix}$,

$k = 0$ is a pole of order 2.

4. There exists an operator with the root system which forms a basis of H but under a different choice of the root vectors the root system of this operator does not form a basis of H (see [*] for an example).

5. The set of complex poles of the Green's function of the Schrödinger operator does not define uniquely the potential if there are bound states [*].

6. If z is a complex pole of order m of $(I+T(k))^{-1}$, $T(k,\varepsilon)$ is compact and analytic in k and ε for $\{|k-z| \leq a, |\varepsilon| < b\}$ and $T(k,0) = T(k)$ then the poles $z(\varepsilon)$ of $(I+T(k,\varepsilon))^{-1}$ can have a branch point at $\varepsilon = 0$ and ord $z(\varepsilon) \leq m$. Moreover $z(\varepsilon)$ can be represented by Puiseux series, i.e. by a series in powers of $\varepsilon^{1/r}$, where r is some integer (see [*] for details).

7. The multiplicity of the complex poles is not equal to the order of zeros of eigenvalues, generally speaking.

It was proved in [6.72] (see also [6.68]) that the set of complex poles coincide with the set of complex zeros of the eigenvalues of certain integral equations. In the case we are concerned with in this paper one can have in mind the eigenvalues of the equation $[I+T(k)]\, u_j = \lambda_j(k)u_j$, $j = 1, 2, \ldots$. It was an open question whether the orders of the zeros of $\lambda_j(k)$ are equal to the multiplicities of the corresponding poles. We show by presenting an example that this is not so in general. Let us take as $I + T(k)$ a finite dimensional operator with the following matrix

$A(k) = \begin{pmatrix} \lambda(k) & 1 \\ 0 & \lambda(k) \end{pmatrix}$. We have $\lambda_j(k) = \lambda(k)$, $A^{-1}(k) = \begin{pmatrix} \lambda^{-1}(k) & -\lambda^{-2}(k) \\ 0 & \lambda^{-1}(k) \end{pmatrix}$.

If $\lambda(z) = 0$ and m is the order of the zero, then z is the pole of $A^{-1}(k)$ of multiplicity $2m$. It is clear from this example that the order of zeros of the eigenvalues will coincide with the multiplicity of the corresponding poles iff $A(k)$ is diagonalizable, that is $A(k)$ has no root vectors. This example is sufficiently general because for a compact T the eigenvalues $\lambda_j \neq -1$ have finite algebraic multiplicites and the corresponding root spaces reduce $I + T(k)$, so that in the root spaces $I + T(k)$ is a matrix operator.

8. Using the ideas given in [6.68] the author proved convergence of the T-matrix approach in scattering theory, widely used in practice.

9. A variational principle for complex poles

In section 3 it was mentioned that the complex poles of the Green function occur at the complex points k at which the homogeneous equation (2) has a nontrivial solution. Let H_q denote the Sobolev space $W_2^q(\Gamma)$, and $|f|_q$ denote the norm in H_q. Consider the variational principle $F(f) = |Af|_1^2 = \min$, $|f|_0 = 1$. If $\{f_j\}$ is a basis of $H = H_0$, and $f^{(n)} = \sum_{j=1}^{n} c_j f_j$, then the problem $F(f^{(n)}) = \min$, $|f^{(n)}|_0 = 1$ yields: $\sum_{m=1}^{n} a_{jm}(k)c_m = 0$, $1 \leq j \leq n$, $\sum_{j=1}^{n} |c_j|^2 > 0$. Thus (*) $\det a_{jm}(k) = 0$. Let $k_s^{(n)}$ be the complex roots of (*). Then it can be proved that the set of the complex limit points $\{k_s\}$ of the set $\{k_s^{(n)}\}$ coincides with the set of the complex poles of the Green function. This a new result. The functional $F(f)$ is real valued in contrast with the functional $K(u)$ in (10').

Problems

1) Is it true that the root systems of $A(k)$, $T(k)$ form a Riesz basis of H? It is proved that these systems form a Riesz basis with brackets (see [6.68] for a proof and definitions). The author thinks that the answer is no.

2) Is there a relation between the order of a complex pole and the multiplicity of the zeros of $\lambda_n(k)$?

3) Can the scatterer be uniquely identified by the set of complex poles of the corresponding Green's function?

4) Prove that there are infinitely many complex poles k_j with $\mathrm{Re}\, k_j \neq 0$ (in diffraction problems and noncentral potential scattering).

5) Are the complex poles of the Green's function of the exterior Dirichlet or Neumann Laplacian simple?

6) Make numerical experiments in the calculation of the complex poles.

7) Prove convergence of the numerical procedure for calculation of the complex poles suggested in [6.74].

8) Find a theoretical approach optimal in some sense to approximate a function $f(t)$ by the functions of the form $f_N = \sum_{j=1}^{N} \sum_{m=1}^{m_j} \exp(-ik_j t)t^{m-1} c_{mj}$. Here the numbers c_{mj}, m_j, k_j are to be found so that f_N will approximate $f(t)$ in some

optimal way. Currently some methods (e.g. Prony method) are used in practice, but they are not optimal. This problem seems to be of general interest (optimal harmonic analysis in complex domain).

9) When can SEM in the form of (10) be justified?

Conclusion

We hope that it was shown in this paper that:

1) EEM is justified (in the generalized form of expansion in root vectors).

2) SEM is justified in the asymptotic form (9).

3) Numerical projection method for calculation of the complex poles is justified.

4) There are many interesting and difficult open problems in the field.

5) Numerical results and experiments are desirable.

Reference

(*) Ramm, A.G., Mathematical foundations of the singularity and eigenmode expansion methods, J. Math. Anal. Appl. (1981).

APPENDIX III

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 86, 562–591 (1982)

Mathematical Foundations of the Singularity and Eigenmode Expansion Methods (SEM and EEM)

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Contents.

INTRODUCTION

This paper is a summary of the invited lecture given by the author at the meeting on mathematical foundations of SEM in November 1980. There were engineers, physicists, and mathematicians at this meeting. Thus this paper was written for readers with various interests and backgrounds. The questions under consideration are of practical interest in the fields where wave propagation and scattering are of importance, that is, in the fields of unlimited diversity. On the other hand, the underlying mathematical theory is deep and relatively new. The mathematical machinery includes the spectral theory of nonselfadjoint operators and pseudo-differential equations on compact manifolds. The mathematical results of use in the EEM and SEM were obtained relatively recently. The author tried to present some of the results and their applications as simply as he could. Whether he succeeded, the reader will tell.

* Supported by AFOSR 800204.

1. STATEMENT OF THE SEM AND EEM

1.0.

SEM and EEM were widely used by physicists and electrical engineers during the last decade [3, 10, 30, 34, 35]. Their mathematical analysis was started in [25, 26] (see also [28, 31]).

1.1.

Let us formulate the EEM. Consider the problem

$$(\nabla^2 + k^2)\, u = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$u\,|_\Gamma = f, \tag{1.2}$$

$$r(\partial u/\partial r - iku) \to 0 \qquad \text{as} \quad r \to \infty, \tag{1.3}$$

where $k > 0$, $r = |x|$, Ω is an exterior domain with a smooth boundary Γ. Let us look for a solution of (1.1)–(1.3) of the form

$$u = \int_\Gamma G_0(x, s, k)\, g(s)\, ds, \tag{1.4}$$

where

$$G_0(x, y, k) = \frac{\exp(ik\,|x - y|)}{4\pi\,|x - y|} \tag{1.5}$$

and $g(s)$ is an unknown function. Function (1.4) satisfies (1.1) and (1.3). Substituting (1.4) into (1.2) we get the following equation for g:

$$Ag = f, \qquad Ag = \int_\Gamma G_0(s, s', k)\, g(s')\, ds', \qquad s \in \Gamma. \tag{1.6}$$

Equation (1.6) is an integral equation of the first kind. In Section 3 we will study this equation. At this moment we restrict ourselves by describing the EEM. Suppose that the operator A in (1.6) has eigenvectors

$$Af_j = \lambda_j f_j, \qquad |\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \tag{1.7}$$

and the set $\{f_j\}$ forms a basis of $H = L^2(\Gamma)$. This means that any element $f \in H$ can be uniquely represented by a convergent in H series

$$f = \sum_{j=1}^{\infty} c_j f_j. \tag{1.8}$$

If this assumption is true, then one can look for a solution of (1.6) of the form

$$g = \sum_{j=1}^{\infty} g_j f_j, \tag{1.9}$$

substitute (1.9) and (1.8) into (1.6), and find the unknown coefficients $g_j : g_j = \lambda_j^{-1} c_j$. Thus

$$g = \sum_{j=1}^{\infty} \lambda_j^{-1} c_j f_j \tag{1.10}$$

is the solution of (1.6). This was the argument used in [3, 10, 35]. The above method for solving Eq. (1.6) was called EEM. It was pointed out in [25] that the operator A in (1.6) is nonselfadjoint and therefore it is not obvious that A has eigenvalues. It is even less clear that the eigenvectors of A form a basis of H. Indeed, even in a finite-dimensional space a linear operator (a matrix) can have a set of eigenvectors which does not form a basis. For example, if $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a matrix of an operator on $\mathbb{R}^2$ (two-dimensional Euclidean space), then A has only one eigenvector and this vector certainly does not form a basis of $\mathbb{R}^2$. Nevertheless, it is known that a root system of a linear operator on $\mathbb{R}^n$ forms a basis of $\mathbb{R}^n$. By a root system of a linear operator A we mean the union of the root vector of A. To construct the root vectors of A we take an eigenvalue λ_j and a corresponding eigenvector f_j and consider the equations

$$Af_j^{(1)} - \lambda_j f_j^{(1)} = f_j, \ldots, Af_j^{(r)} - \lambda_j f_j^{(r)} = f_j^{(r-1)}. \tag{1.11}$$

If these equations are solvable but the equation $Af_j^{(r+1)} - \lambda_j f_j^{(r+1)} = f_j^{(r)}$ is not solvable, then the set $(f_j, f_j^{(1)}, \ldots, f_j^{(r)})$ is called the Jordan chain associated with the pair (λ_j, f_j), $r+1$ is the length of this chain, and $f_1^{(1)}, \ldots, f_j^{(r)}$ are called the root vectors of A. If A is the compact operator on a Hilbert space, the definition is the same. It is known [7] that a compact linear operator on a Hilbert space has a discrete spectrum with the only limit point $\lambda = 0$ and the length of any Jordan chain associated with a pair (λ_j, f_j), $\lambda_j \neq 0$ is finite. In a finite-dimensional space $\mathbb{R}^n$ the root system of every linear operator forms a basis of $\mathbb{R}^n$. Unfortunatly this is not true in the infinite-dimensional Hilbert space. For example the Volterra operator $Vf = \int_0^x f\,dt$ on $H = L^2[0, 1]$ has no eigenvalues. Thus, we face the following basic problems:

(1) *When does a nonselfadjoint operator A on H have a root system which forms a basis of H?*

(2) *When does the set of eigenvectors of A form a basis of H?*

It is clear that the EEM as described by formula (1.10) is not valid generally speaking because one should take into account the root vectors when writing the series for g and f. This will not make the calculation much more difficult, as we will show in Section 3. Therefore from now on we will mean by EEM the solution of (1.6) by means of expansion in series in root vectors of A. Both questions (1) and (2) will be discussed in Section 3. It should be mentioned that the specific form of the boundary value problem (1.1)–(1.3) does not play any significant role. We can treat by the same methods the Neumann or the third boundary value problems. What is essential is that the problem in the 3-dimensional unbounded domain Ω is reduced to an equation on 2-dimensional compact manifold (surface Γ).

1.2.

We now pass over to the SEM. Let us consider the problem

$$u_{tt} = \nabla^2 u, \qquad t \geqslant 0, \quad x \in \Omega, \tag{1.12}$$

$$u|_\Gamma = 0, \tag{1.13}$$

$$u|_{t=0} = 0, \qquad u_t|_{t=0} = f(x). \tag{1.14}$$

If we define

$$v(x,k) = \int_0^\infty \exp(ikt)\, u(x,t)\, dt, \tag{1.15}$$

then

$$(\nabla^2 + k^2)\, v = -f, \tag{1.16}$$

$$v|_\Gamma = 0, \tag{1.17}$$

$$r(\partial v/\partial r - ikv) \to 0, \qquad r \to \infty. \tag{1.18}$$

Thus

$$v(x,k) = \int_\Omega G(x,y,k)\, f(y)\, dy, \tag{1.19}$$

where G is the Green's function for problem (1.16)–(1.18). We have

$$G = G_0 - \int_\Gamma G_0(x,s,k) \frac{\partial G(s,y,k)}{\partial n_s}\, ds. \tag{1.20}$$

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and for $\mu = \partial G(s, y, k)/\partial n_s$ it is easy to get the equation

$$|I + T(k)|\,\mu = 2\frac{\partial G_0}{\partial n_s},$$

$$T(k)\,\mu \equiv \int_\Gamma \frac{\partial}{\partial n_s} \frac{\exp(ik|s-s'|)}{2\pi|s-s'|}\,\mu(s')\,ds'. \qquad (1.21)$$

From (1.21) it follows that μ is a meromorphic function of k on the whole complex plane k and from this and (1.20) we conclude that $G(x, y, k)$ can be analytically continued as a meromorphic function of k on the whole complex plane k. Moreover the residues of $G(x, y, k)$ (and $\mu(s, y, k)$) are kernels of operators of finite rank (degenerate kernels). This conclusion is an immediate corollary to the following:

PROPOSITION 1.1. *Let $T(k)$ be an analytic compact operator function on H for $k \in \Delta$, where Δ is a connected open set in the complex plane. If $I + T(k)$ is invertible at some point $k_0 \in \Delta$, then $(I + T(k))^{-1}$ is finite-meromorphic in Δ.*

Remark. Finite-meromorphic means that the Laurent coefficients are operators of finite rank. Though the proposition is well known we will give a short proof in Section 3 for the sake of completeness.

From (1.15) it follows that

$$u(x, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(-ikt)\, v(x, k)\, dk. \qquad (1.22)$$

The function $v(x, k)$ is analytic in the half plane $\operatorname{Im} k \geqslant 0$ and meromorphic in $\operatorname{Im} k < 0$.

Let us introduce the following three conditions:

$$v \text{ is meromorphic (and analytic in } \operatorname{Im} k \geqslant 0), \qquad (1.23)$$

$$|v| \leqslant c(b)(1 + |k|)^{-a}, \quad a > \tfrac{1}{2}, \quad |\operatorname{Re} k| \to \infty, \quad \operatorname{Im} k = b, \qquad (1.24)$$

where b is an arbitrary constant,

$$|\operatorname{Im} k_j| \to \infty \qquad as \quad j \to \infty, \qquad (1.25)$$

where $\{k_j\}$ are the poles of v ordered so that $|\operatorname{Im} k_1| \leqslant |\operatorname{Im} k_2| \leqslant \cdots$

In (1.23) we can assume that v has a finite number of poles in $\operatorname{Im} k \geqslant 0$. Also assumption (1.24) can be relaxed: we can assume that a is an arbitrary fixed number. We will not, however, discuss these possibilities here. The assumption $a > \frac{1}{2}$ guarantees that the integral in (1.22) converges in L^2. (1.24) $\Rightarrow$ (1.25).

Using (1.23)–(1.25) and moving the contour of integration in (1.22) down, we get

$$u(x, t) = \sum_{j=1}^{N} c_j(x, t)\, e^{-ik_j t} + o(e^{-|\operatorname{Im} k_N| t}), \qquad t \to +\infty. \qquad (1.26)$$

Here

$$c_j(x, t)\, e^{-ik_j t} = \operatorname*{Res}_{k=k_j} v(x, k)\, e^{-ikt}, \qquad c_j(x, t) = O(t^{m_j - 1}) \qquad (1.27)$$

and m_j is the order of the pole k_j. If and only if all the poles are simple, then $c_j(x, t) = c_j(x)$. We have proved

PROPOSITION 1.2. *Conditions* (1.23)–(1.25) *are sufficient for the "asymptotic"* SEM.

By asymptotic SEM we mean formula (1.26). By SEM we mean the formula

$$u(x, t) = \sum_{j=1}^{\infty} c_j(x, t) \exp(-ik_j t), \qquad (1.28)$$

where series (1.28) converges uniformly in x and t running through bounded domains. In Section 3 we will show that conditions (1.23)–(1.25) can be verified under relatively general assumptions about the scatterer. Therefore the asymptotic SEM in the form (1.26) can be established. But SEM in the form (1.28) does not seem to be established, even under very restrictive assumptions about the scatterer. It is an open question:

(3) *When does* (1.28) *hold?*

2. DISCUSSION OF THE RELATED QUESTIONS

2.1. *Interpretation of the* EEM *as an Eigenoscillation Method with Spectral Parameter in Boundary Conditions*

As we pointed out in Section 1 the mathematical idea behind the EEM can be formulated as follows: We substitute the boundary value problem in the exterior 3-dimensional domain by an integral equation over 2-dimensional compact manifold. In Section 3 we will show that this integral equation is a pseudo-differential equation with an elliptic pseudo-differential operator. These terms will be explained in Section 3. Here we want to show

that there is a possibility of a physical interpretation of the EEM. Indeed, let function (1.4) satisfy the equations

$$(\nabla^2 + k^2)\,u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \Gamma, \tag{2.1}$$

$$u^+ = u^-, \qquad u = \lambda[(\partial u/\partial n)^+ - (\partial u/\partial n)^-] \quad \text{and} \quad (1.3), \tag{2.2}$$

where n is the outer unit normal to Γ and $+$ $(-)$ denote the limit value on Γ from inside (outside) of Γ. Since $u|_\Gamma = Ag$, where A is defined in (1.6), and $(\partial u/\partial n)^+ - (\partial u/\partial n)^- = g$, the second equality in (2.2) is equivalent to equation $Ag = g$. Thus expansion (1.9) is the expansion in eigenfunction of problem (2.1)–(2.2), where the spectral parameter λ is in boundary condition. This parameter differs from the usual frequency parameter in the classical approach.

2.2. *Complex Poles of Green's Function (Resonances), Existence, Multiplicity, Calculation, Stability, and Asymptotic Formulas.*

In Section 1 we saw that the complex poles k_j of the Green's function $G(x, y, k)$ are important in SEM (see (1.26)). It seems to be an open question whether the simplicity of the complex poles of G is equivalent to the absence of the root vectors of the operator $T(k)$ defined in (1.21). In [14] it was proved that there are infinitely many purely imaginary poles $i\tau_n$, $\tau_n \to -\infty$, but it is still an open question whether there are infinitely many complex poles of G off the imaginary axis. For one-dimensional potential scattering (the Schrödinger equation on the semiaxis) it was proved that the Green's function has infinitely many complex poles (see [17]). This proof cannot be carried out for three-dimensional potential scattering, because it uses essentially the expression of the Green's function in terms of two linearly independent solutions of the Schrödinger equation. It would be interesting to work out a new proof which covers the three-dimensional case. It was proved in [19–21, 23] that the Green's function of the Laplace operator of exterior boundary value problems can be analytically continued on the whole complex plane of k as a meromorphic function. In [22] it was proved that (1.25) holds for the Schrödinger operators with compactly supported potentials. In [15] it was proved for the Laplace operator in the exterior domain and in [24] for the Schrödinger operator in the exterior domain. In [24], estimate (1.24) was introduced and established. Thus we have

PROPOSITION 2.1. *The asymptotic* SEM *in the form* (1.26) *holds for smooth star-like scatterers.*[1]

A body D is called star-like if there exists a point x_0 inside D such that every point on the boundary Γ of D can be seen from x_0 (star-like = star-shaped).

[1] More generally, for nontrapping scatterers [16].

It is an open question whether the complex poles k_j are simple. Engineers and physicists conjectured that this is the case, but no conclusive arguments were given. For the spherical and linear obstacles the poles are simple, but this is due to the fact that the operator A in (1.6) is normal (i.e., $A^*A = AA^*$) if Γ is a sphere or a line [25, 31]. To show that *there can be multiple poles of the Green's function of the third boundary value problem*, consider the following.

EXAMPLE. Let

$$(\nabla^2 + k^2)\,u = 0 \quad \text{in} \quad \Omega = \{x : |x| \equiv r \geqslant 1\}, \quad x \in \mathbb{R}^3, \quad k > 0, \tag{2.3}$$

$$(\partial u/\partial r - 2u)|_{r=1} = \cos\theta \quad \text{and} \quad (1.3) \text{ holds}. \tag{2.4}$$

It is easy to find the solution to this problem:

$$u = \frac{-ik \cdot \text{const}\, r^{-1/2} H_{3/2}(kr) \cos\theta}{e^{ik}(k^2 + 4ki - 4)} \tag{2.5}$$

Thus, $k = -2i$ is a pole of order 2. Note that for $k > 0$ problem (2.3)–(2.4) has a unique solution so that the existence of the multiple pole cannot be explained by the presence of active impedance sheet on Γ: the boundary condition (2.4) is passive in the sense that for $k > 0$ the homogeneous problem (2.3)–(2.4) has only the trivial solution $u = 0$. — A22 —

How does one calculate the complex poles? Are they stable under small perturbations of Γ? These questions were answered in [25, 28, 31]. *We describe three different approaches* given in [31]. *The first approach is a general projection method.* It was introduced for calculation of the poles in [25]. The complex poles of G are the points at which the operator $I + T(k)$ (see (1.21)) is not invertible (has a nontrivial null space). Let $\{h_j\}$ be a basis of $H = L^2(\Gamma)$, $F_n = \sum_{j=1}^n c_j h_j$. We substitute the equation $[I + T(k)]\,F = 0$ by the equation $P_n[I + T(k)]\,P_n F = 0$, where P_n is the projection on the linear span of $\{h_1, \ldots, h_n\}$. This leads to the linear system

$$\sum_{j=1}^{n} b_{ij}(k)\, c_j = 0, \qquad 1 \leqslant i \leqslant n, \qquad b_{ij} = ([I + T(k)]\, h_i, h_j). \tag{2.6}$$

where $(\cdot, \cdot)$ denotes the inner product in $H = L^2(\Gamma)$. System (2.6) has a nontrivial solution iff

$$\det b_{ij}(k) = 0. \tag{2.7}$$

In the left-hand side of (2.7) we have an entire function of k. Let $k_m^{(n)}$ be its zeros.

Proposition 2.2. *The set of $k_m = \lim_{n\to\infty} k_m^{(n)}$ coincides with the set of the points at which $I + T(k)$ is not invertible, that is with the union of the set of the complex poles of the Green's function G defined in* (1.20) *and the spectrum of the interior Neumann problem.*

This proposition justifies the current numerical method widely used by engineers for calculation of the complex poles. Its proof is given in [25] and has an interesting by-product:

Proposition 2.3. *The complex poles depend continuously on the scatterer.*

This can be formulated in more detail as follows. Let

$$x_j = x_j(s_1, s_2), \qquad 1 \leqslant j \leqslant 3, \quad 0 \leqslant s_1, \quad s_2 \leqslant 1,$$
$$x_j \in C^2, \quad s = (s_1, s_2)$$

be a parametric equation of the surface Γ, and $x_j(\varepsilon) = x_j(s) + \varepsilon y_j(s)$, $y_j \in C^2$, where $\varepsilon > 0$ is a small parameter, be a parametric equation of the perturbed surface Γ_ε. Let $k_j(k_j(\varepsilon))$ be the complex poles of the Green's function $G(G_\varepsilon)$. Then $k_j(\varepsilon) \to k_j$ as $\varepsilon \to 0$ uniformly for $|k_j| \leqslant R$, where $R > 0$ is an arbitrarily large fixed number. For a detailed proof see [31].

The second approach to the calculation of the poles is based on variational principle. Let us consider the set of functions which have the representation

$$u(x,k) = r^{-1}\exp(ikr)\sum_{j=0}^{\infty} f_j(n,k)\, r^{-j},$$
$$n = x|x|^{-1}, \quad r = |x|, \tag{2.8}$$

$f_0 \not\equiv 0$. The solutions to the Helmholtz equation in an exterior domain satisfying the radiation condition for $k > 0$ satisfy (2.8). It can be proved [30] that if $u(x, k_1)$ and $v(x, k_2)$ belong to the above set, $\mathrm{Re}(k_1 + k_2) \neq 0$, $\pi < \arg k_m < 2\pi$, $m = 1, 2$, then the following limit exists,

$$\langle u, v\rangle \equiv \lim_{\varepsilon\to+0} \int \exp(-\varepsilon r \ln r)\, uv\, dx, \qquad \int = \int_\Omega, \tag{2.9}$$

and the complex poles of G (defined in 1.19) are the stationary values of the functional

$$k^2 = \mathrm{st}\,\frac{\langle \nabla u, \nabla u\rangle}{\langle u, u\rangle}. \tag{2.10}$$

The admissible functions in (2.10) should satisfy (2.8) and vanish on Γ. Some choice of the basis functions for principle (2.10) is suggested in [30]. *The third approach to the calculation of the complex poles is based on the following statement* which was proved in [25] (see also [28]).

Proposition 2.3′. *The set of the complex poles of G coincide with the set of the complex zeros of the functions $\lambda_n(k)$, where $\lambda_n(k)$ are the eigenvalues of the operator $A(k)$ defined in* (1.6). *The set of all zeros of the functions $\lambda_n(k)$, $n = 1, 2, \ldots$, is the union of the set of complex poles of G and the set of eigenvalues of the Dirichlet Laplacian in the interior domain D with boundary Γ.*

For a proof see [31].

Remark [31]. The set of complex poles of G can be also found as the set of the complex roots of the equations $\mu_n(k) = -1$, $n = 1, 2, \ldots$, where $\mu_n(k)$ are he eigenvalues of the operator $T(k)$ defined in (1.21).

According to Proposition 2.3′ we can calculate the complex poles by calculating the functions $\lambda_n(k)$ and finding their complex zeros. The eigenvalues $\lambda_n(k)$ can be found by means of the projection method. *In Section 3 we give a new variational principle for the spectrum of a compact nonselfadjoint linear operator on a Hilbert space.*

2.3. *Mittag-Leffler Representation*

From (1.20) and (1.21) it follows that the poles of G coincide with the poles of the operator $(I + T(k))^{-1}$. This operator is a meromorphic function. One can apply the Mittag-Leffler representation to this function. Since in the engineering literature [35] the Mittag-Leffler theorem was used not quite accurately, we give the statement of the theorem here and discuss the difficulties of its application to our problem (representation of $(I + T(k))^{-1}$).

Proposition 2.4. *Let $f(k)$ be a meromorphic function on the whole complex plane k and $|f(k)| \leqslant c|k|^p$, $k \in C_n$, where C_n is a proper system of contours and $p \geqslant 0$ is an integer. Let us assume (without loss of generality) that $k = 0$ is not a pole of f. Then*

$$f(k) = h(k) + \sum_{n=1}^{\infty} [g_n(k) - h_n(k)], \tag{2.11}$$

where

$$h(k) = \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j!} k^j, \qquad h_n(k) = \sum_{j=0}^{p} \frac{g_n^{(j)}(0)}{j!} k^j, \tag{2.12}$$

and $g_n(k)$ is the principal part of $f(k)$ at the pole k_n.

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Remark. A proper system of contours $\{C_n\}$ is a system of closed curves such that

(1) $k=0$ lies inside C_n,

(2) $D_n \subset D_{n+1}$, where D_n is the domain inside C_n,

(3) $d_n = \operatorname{dist}(0, C_n) \to \infty$ as $n \to \infty$ and $d_n^{-1}|C_n| \leqslant c$, $c = \text{const}$, where $|C_n|$ is the length of C_n.

A more general statement is the following proposition.

PROPOSITION 2.4′. *Let $f(k)$ be a meromorphic function. There exists a sequence of integers $p_1, \ldots, p_n, \ldots$ such that* (2.11) *holds with*

$$h_n(k) = \sum_{j=0}^{p_n} \frac{g_n^{(j)}(0)}{j!} k^j. \tag{2.13}$$

Remark. We do not know how fast the numbers p_n in (2.13) grow and therefore it seems impossible to use Proposition 2.4′ for numerical calculations. The estimate $\|(I+T(k))^{-1}\| \leqslant c|k|^p$, $k \in C_n$, is not known, so that (2.11) is also difficult to apply. Even in the case when the poles are simple (so that $g_n(k) = c_n/(k-k_n)$) we do not know $h_n(k)$ and therefore can not use (2.11). In the engineering literature, sometimes the formula

$$f(k) = \sum_{n=1}^{\infty} \frac{c_n}{k-k_n} \tag{*}$$

was used. This is not correct because the series $\sum_{n=1}^{\infty} g_n$ does not converge in general. Even if $p=0$, formula (2.11) takes the form

$$f(k) = f(0) + \sum_{n=0}^{\infty} \left(\frac{c_n}{k-k_n} + \frac{c_n}{k_n} \right), \tag{2.14}$$

which differs from (*).

2.4. *Perturbation of Complex Poles (Resonances)*

Consider the problem in a general setting. Let $T(k)$ be an analytic compact operator function such that $I+T(k)$ is invertible for some k. Then $(I+T(k))^{-1}$ is finite meromorphic (Proposition 1.1). Let z be a pole of order m of the function $(I+T(k))^{-1}$. Let $T(k,\varepsilon)$ be a compact operator which is analytic on $\{|k-z|<a, |\varepsilon|<b\}$ and such that $T(k,0)=T(k)$. We want to study the poles of $(I+T(k,\varepsilon))^{-1}$ as functions of ε. *Our conclusion is as follows: Under a perturbation depending analytically on ε the multiplicity of the pole z cannot increase. It can decrease and the pole $z(\varepsilon)$ of $(I+T(k,\varepsilon))^{-1}$ can have a branch point $\varepsilon=0$ as a function of ε.* It can be

represented by Puiseux series, i.e., by a series in the powers of $\varepsilon^{1/r}$, where r is some integer. A proof is given in Section 3.

2.5. *Asympototics of Resonances*

In this section we give some asymptotic formulas for the large complex poles nearest to the real axis. Consider the exterior domain Ω and assume that its boundary Γ is smooth and convex and its Gaussian curvature is strictly positive. In the two-dimensional case the following formula for the complex poles of G can be obtained by the method of geometrical optics,

$$k_{pq} \approx \frac{2\pi q}{|\Gamma|}\left(1 - \frac{c\xi_p}{(2\pi q)^{2/3}}\right), \qquad q \gg 1, \tag{2.15}$$

where $c = \text{const}$ depends only on the geometry of Γ, $|\Gamma|$ is the length of Γ, $\xi_p = t_p \exp(i\pi/3)$, and $t_p < 0$ are the zeros of the Airy function $v(t) = \pi^{-1/2}\int_0^\infty \cos(ty + y^3/3)\,dy$, and p is an integer small in comparison with q.

From (2.15) it follows that

$$|\operatorname{Im} k_{pq}| = O(|\operatorname{Re} k_{pq}|^{1/3}), \qquad q \gg 1. \tag{2.16}$$

This estimate can be verified for a circle by direct calculation of the complex poles.

If the boundary Γ is not smooth then instead of (2.16) *one can get*

$$|\operatorname{Im} k_{pq}| = O(\ln|\operatorname{Re} k_{pq}|), \qquad q \gg 1. \tag{2.17}$$

Let us explain this by taking a polygon as Γ. The field diffracted by a wedge is proportional to $\exp(ikr - \frac{1}{2}\ln(kr))$. Consider a ray having passed once around the polygon. The phase of the field at the point of destination is $ik|\Gamma| - \frac{1}{2}\ln|k^n|$ + terms which do not depend on k. We assume that the polygon has n sides. In order that the field amplitude conserves, one requires that the quantization condition

$$ik|\Gamma| - (n/2)\ln k = 2\pi qi, \tag{2.18}$$

be satisfied, where q is an integer. For $q \gg 1$, one gets

$$k_q \approx \frac{2\pi q}{|\Gamma|} - \frac{in \ln(2\pi q)}{2|\Gamma|}, \qquad q \gg 1. \tag{2.19}$$

Formula (2.17) follows from (2.19). For the exterior 3-dimensional domain with a smooth convex boundary with positive Gaussian curvature a similar result to (2.15) can be obtained. An additional difficulty in the 3-dimensional problem consists in finding a closed elliptic geodesic $\mathscr{L}$ on Γ. Let s be the

length along $\mathscr{L}$, v be the coordinate measured along the geodesic orthogonal to $\mathscr{L}$, $K(s)$ be the Gaussian curvature at the points on $\mathscr{L}$, and $T = |\mathscr{L}|$ be the length of $\mathscr{L}$. Consider the equation

$$d^2v/ds^2 + K(s)\,v = 0, \qquad K(s+T) = K(s), \qquad -\infty < s < \infty. \qquad (*)$$

The geodesic $\mathscr{L}$ is called elliptic if Eq. (*) is stable in the sense of Liapunov. Formulas of this section can be found in [11]. In [1, 2, 18, 33] some asymptotic formulas for the Green's function as $k \to +\infty$, $\operatorname{Im} k = 0$ are given, but they seem to be of no use in calculating the complex poles. The reason is that the formulas give an expression for the Green's function in terms of exponential functions (geometrical optics) and this expression has no poles.

2.6. *Nonsmooth Boundaries*

If we want to apply Proposition 1.1 to the problem with a nonsmooth boundary Γ (for example, surface with conical points or edges) we face the following difficulty: the operator $T(k)$ defined in (1.21) is not compact if Γ is not smooth. Potential theory for domains with nonsmooth boundaries was studied in [4]. In this section we will show how to handle the above difficulty. To this end let us first define an essential norm of a linear operator T: $|T|_{ess} = \inf_{Q \in K} \|T - Q\|$, where K is the set of compact operators. Assume that $|T|_{ess} < 1$. Consider the equation $(I+T)g = f$ in a Hilbert space. By our assumption we can write $T = S + Q$, where Q is compact and $\|S\| < 1$. Therefore our equation is equivalent to the equation $(I+Q_1)g = f_1$, $Q_1 = (I+S)^{-1}Q$, $f_1 = (I+S)^{-1}f$, where Q_1 is compact and $(I+S)^{-1}$ is a bijection of H because $\|S\| < 1$ (bijection is a continuous map onto H which has continuous inverse). Therefore, the equation $(I+T)g = f$ with a noncompact operator T with $|T|_{ess} < 1$ is equivalent to the equation with compact operator. This argument shows that the following generalization of Proposition 1.1 holds.

PROPOSITION 2.5. *If $T(k)$ can be represented in the form $T(k) = T + Q(k)$, where $Q(k)$ is analytic and compact, $|T|_{ess} < 1$ and $I + T(k)$ is invertible at some point, then $(I + T(k))^{-1}$ is finite-meromorphic.*

In order to apply this proposition, we use the result from [4] which says that $|T(0)|_{ess} < 1$ if the surface Γ is piecewise smooth, has no cusps and its irregular points are conical or the edge of the wedge. (In fact, in [4] much more general results are given, but they are of no interest to us at this moment. When the surface has cusps we are in trouble, otherwise the theory given in Section 1 holds.) We can write $T(k) = T(0) + T(k) - T(0)$. If $Q(k) = T(k) - T(0)$, then $Q(k)$ is analytic and compact and we can use Proposition 2.5. This argument shows that the meromorphic nature of the Green's function holds also in case of nonsmooth boundaries without cusps.

2.7. *Asymptotics of Resonant States. Their Orthogonality*

Let $a - ib$, $b > 0$, be a complex pole of G. A resonant state is a solution to the problem

$$(\nabla^2 + k^2)\,u = 0 \text{ in } \Omega,\ k = a - ib, \qquad b > 0; \quad u|_\Gamma = 0, \qquad (2.20)$$

satisfying (2.8).

Remark. The radiation condition cannot be used for the statement of the problem of finding the resonant tate and the corresponding complex poles of the Geen's function. Indeed, the problem $(\nabla^2 + k^2)\,u = 0$ in $\mathbb{R}^3$, where u satisfies (1.3), has a nontrivial solution for any k with $\operatorname{Im} k < 0$. For example, if f is a smooth compactly supported function then

$$u = \int \frac{\exp(ik|x-y|)}{|x-y|} f(y)\,dy - \int \frac{\exp(-ik|x-y|)}{|x-y|} f(y)\,dy$$

is such a solution because the second integral is $O(\exp(-|\operatorname{Im} k|\,r))$ as $r \to \infty$ and does not change the radiation condition (1.3).

The solution of (2.20) satisfies the estimate $u = O(r^{-1}\exp(br))$. We want to answer the folowing question: What can be said about u if $u = o(r^{-1}\exp(br))$ as $r \to \infty$? The answer is: $u \equiv 0$ in this case. In order to prove this statement for $a = 0$ consider the function $v = r\exp(-ikr)\,u$, $k = -ib$. By the assumption $v = o(1)$ as $r \to \infty$. It is easy to see that

$$-v'' - r^{-2}\Delta^* v - 2ikv' = 0 \qquad \text{for} \quad r > R_0, \qquad (*)$$

where Δ^* is the angular part of the Laplacian and R_0 is the radius of a sphere containing Γ. Multiplying (*) by v in $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$, integrating in r over (R, ∞) and taking the real part, we get

$$0 = \int_R^\infty |v'|^2\,dr + \int_R^\infty (-\Delta^* v, v)\,r^{-2}\,dr + \frac{1}{2}\frac{d|v|^2}{dr}\bigg|_{r=R} + b|v|^2|_{r=R}.$$

Thus $(d/dr)|v|^2 + 2b|v|^2 < 0$, $r > R_0$. This implies that $|v|^2 = O(\exp(-2br))$, $|v| = O(\exp(-br))$, and $u = O(1/r)$. Therefore u can be represented by the Green's formula

$$u = \int_\Gamma \bar{G}_0 \frac{\partial u}{\partial N}\,ds, \qquad \bar{G}_0 = \exp(-ikr)/(4\pi r), \qquad k = -ib, \qquad b > 0.$$

Therefore $u = O(\exp(-br))$, $u \in L^2(\Omega)$, and $-k^2$ is an eigenvalue of the Dirichlet Laplacian in Ω. Since this operator has no eigenvalues we conclude that $u \equiv 0$.

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If $a \neq 0$, then by the assumption, $f_0 = 0$ in (2.8). One can verify that from Eq. (2.20) it follows that $f_{j+1} = [i/2(j+1)k][\frac{1}{4} - (j+\frac{1}{2})^2 - \Delta^*]f_j$, $j \geqslant 0$, where f_j are defined in (2.8). Thus $f_0 = 0 \Rightarrow f_j = 0 \Rightarrow u \equiv 0$. We did not use the boundary condition in this argument.

Another proof can be constructed on Theorem 3 from [11]. Let us answer another question: *In what sense can the resonant states corresponding to different complex poles k_1 and k_2 be considered as orthogonal?*

The answer is $\langle u(x, k_1), u(x, k_2)\rangle = 0$, where the form $\langle \cdot, \cdot \rangle$ was defined in (2.9). For details see [30] and Section 3.

Remark. In the EEM method we can use the symmetry property of the operator A for finding the coefficients in expansion (1.8). Indeed, let us introduce the form $\int fg\,dx \equiv [f, g]$. It is clear that

$$[Af, g] = [f, Ag]. \tag{2.21}$$

If we assume that the eigenvectors f_j of A form a basis of $H = L^2(\Gamma)$, then expansion (1.8) holds and $[f_j, f_m] = 0$ if $j \neq m$. The last statement follows from (2.21): if $Af_j = \lambda_j f_j$, $Af_m = \lambda_m f_m$, $\lambda_j \neq \lambda_m$ then $(\lambda_j - \lambda_m)[f_j, f_m] = [Af_j, f_m] - [f_j, Af_m] = 0$. Thus $[f_j, f_m] = 0$. For $\lambda_j = \lambda_m = \lambda$ one can find linear combinations of the eigenvectors $f_1, \ldots, f_r$ corresponding to λ which are orthogonal with respect to the form $[\cdot, \cdot]$, at least if $[f_j, f_j] \neq 0$, $1 \leqslant j \leqslant r$. This can be used for calculation of the coefficients c_j in (1.8): $c_j = [f, f_j]$.

3. Mathematical Results

3.1. *Justification of* EEM. *Basisness. Convergence of Series in Root Vectors*

We need some definitions. Let $\{h_j\}$ be an orthonormal basis of H, $m_1 < m_2 < \cdots$ be a sequence of integers, $m_j \to \infty$, and H_j be the linear span of the vectors $h_{m_j}, h_{m_j+1}, \ldots, h_{m_{j+1}-1}$. Let $\{f_j\}$ be a complete minimal system in H and F_j be the linear span of $f_{m_j}, \ldots, f_{m_{j+1}-1}$. A system $\{f_j\}$ is called minimal if for any m vector f_m does not belong to the linear span of the remaining vectors $\{f_j\}_{j \neq m}$.

Definition 1. If a linear bijection B exists such that $BH_j = F_j$, $j = 1, 2, \ldots$ then the system $\{f_j\}$ is called a Riesz basis of H with brackets and we write $\{f_j\} \in R_b(H)$.

Let us recall that B is a bijection if it maps continuously and one-to-one H onto H. By basisness we mean the property of a system of vectors to form a basis of H. A system $\{f_j\} \in R_b(H)$ iff there exist $C_2 \geqslant C_1 > 0$ such that for any $f \in H$ the inequality (the analogue of the Bessel inequality) $C_1 \|f\|^2 \leqslant \sum_{j=1}^{\infty} \|P_j f\|^2 \leqslant C_2 \|f\|^2$, holds where P_j is the projection onto F_j.

We write $A \in R_b(H)$ $(A \in R(H))$ if the root system of the linear operator A on H forms a Riesz basis of H with brackets (a Riesz basis of H). Let L be a linear selfadjoint operator on H with discrete spectrum $0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_j \to \infty$ as $j \to \infty$. In this case L^{-1} is compact.

Proposition 3.1. *Let Q be a (nonselfadjoint) linear operator. $D(Q) \supset D(L)$,*

$$\lambda_j = cj^p + O(j^{p_1}) \quad \text{as} \quad j \to \infty, \quad p, c > 0, \quad p_1 < p, \tag{3.1}$$

$$|L^{-a}Q| \leqslant c_a, \quad a < 1; \quad p(1-a) \geqslant 1. \tag{3.2}$$

Then the spectrum $\sigma(A)$ of the operator $A = L + Q$ is discrete, $\sigma(A) \subset \bigcup_{j=1}^{\infty} \{\lambda: |\lambda - \lambda_j| < q|\lambda_j| c_a\}$, where $q > 1$ is an arbitrary number, and $A \in R_b(H)$. If $p(1-a) \geqslant 2$ and $p_1 < p - 1$, then $A \in R(H)$.

A proof of this proposition and some additional information can be found in [31]. Let us show how this proposition can be used in order to prove that $A \in R_b(H)$ and $T(k) \in R_b(H)$, where $A(k)$ and $T(k)$ are defined in (1.6 and (1.21).

We will discuss only $A(k)$ since $T(k)$ can be treated similarly. We have to use some results from the theory of pseudo-differential operators. These results are given in [32]. Let us denote by H^q the Sobolev spaces $W^{2,q}(\Gamma)$. If q is a positive integer, H^q consists, roughly speaking, of functions with q derivatives square integrable over Γ. But H^q is defined for any real q. We say that ord $A = m$ if $A: H^q \to H^{q-m}$, ord A = order of A. By $N(A)$ we denote the null space of A: $N(A) = \{f: Af = 0\}$. We omit some important details and try to explain how to prove that $A(k) \in R_b(H)$.

Let $A = A_0 + A_1$, $A_0 = \operatorname{Re} A$, $A_1 = i \operatorname{Im} A$. The operator $A_0(A_1)$ has the kernel

$$\frac{\cos(k|x-y|)}{4\pi|x-y|} \left(\frac{i\sin(k|x-y|)}{4\pi|x-y|}\right) \quad \text{for} \quad k > 0.$$

We assume for simplicity that A^{-1} and A_0^{-1} exist. This assumption is not essential and can be removed at the cost of some additional technical arguments. Let $A_0^{-1} = L$. The operator L is selfadjoint and it can be shown that ord $L = 1$, $\lambda_j(L) = cj^{1/2} + O(1)$ as $j \to \infty$. We have $A^{-1} = L + Q$, $Q = -(I + LA_1)^{-1} LA_1 L$. The first factor is a bijection and its order is 0. Thus ord $Q = 2$ ord L + ord $A_1 = 2 +$ ord A_1. But A_1 has infinitely smoothing kernel and therefore ord $A_1 = -\infty$. Thus ord $Q = -\infty$. This means that $|L^{-a}Q| < c_a$ for any $a < 1$ (we can take $a < 0$ and $|a|$ as large as we want). From this it follows that conditions (3.1) and (3.2) are satisfied and $A^{-1} \in R_b(H)$. Therefore $A \in R_b(H)$. This argument can be used for complex k also. But in this case the kernels of A_0 and A_1 will be different

and in particular, the kernel of A_1 will not be infinitely smoothing. It can be shown that ord $A_1 = -3$ for complex k.

Now we turn to convergence of the series in root vectors. First we derive some formulas for the coefficients of solution to Eq. (1.6). Let $g = \sum_{j=1}^{\infty} P_j g$, $f = \sum_{j=1}^{\infty} P_j f$, where P_j is the projection on the root space spanned by the root vectors of A corresponding to the pair $(\lambda_j, f^{(0)})$, so that $f_j^{(0)}, \ldots, f_j^{(r_j)}$ is the basis of this root space, $f_j^{(m)}$ are the root vectors. This root space R_j is invariant under the action of A. This means that if $f \in R_j$ then $Af \in R_j$. Therefore $Ag = f$ can be rewritten as $AP_j g = P_j f$, or else

$$\sum_{r=0}^{r_j} g_j^{(r)} A f_j^{(r)} = \sum_{r=0}^{r_j} c_j^{(r)} f_j^{(r)}, \qquad j = 1, 2, \ldots \tag{*}$$

In what follows, we omit the index j for some time. By the definition of root vectors we have $Af^{(r)} - \lambda f^{(r)} = f^{(r-1)}$, $r \geq 1$, $\lambda = \lambda_j$, $Af^{(0)} = \lambda f^{(0)}$. Therefore from (*) it follows that

$$g_j^{(r_j)} = \frac{c_j^{(r_j)}}{\lambda_j}, \qquad g_j^{(m)} = \frac{c_j^{(m)} - g_j^{(m+1)}}{\lambda_j}, \qquad m = r_j - 1, \ldots, 0. \tag{3.3}$$

These recurrent formulas are convenient for the calculation of the coefficients of the expansion of the solution to the equation $Ag = f$ in terms of the root vectors of the operator A. The coefficients $c_j^{(m)}$ corresponding to f are taken to be known. We can rewrite (3.3) as

$$\begin{aligned} g_j^{(r_j)} &= \frac{c_j^{(r_j)}}{\lambda_j}, \qquad g_j^{(r_j-1)} = \frac{c_j^{(r_j-1)}}{\lambda_j} - \frac{c_j^{(r_j)}}{\lambda_j^2}, \\ g_j^{(r_j-2)} &= \frac{c_j^{(r_j-2)}}{\lambda_j} - \frac{c_j^{(r_j-1)}}{\lambda_j^2} + \frac{c_j^{(r_j)}}{\lambda_j^3}, \ldots, \\ g_j^{(m)} &= \frac{c_j^{(m)}}{\lambda_j} - \frac{c_j^{(m+1)}}{\lambda_j^2} + \frac{c_j^{(m+2)}}{\lambda_j^3} + \cdots + \frac{(-1)^{r_j-m} c_j^{(r_j)}}{\lambda_j^{r_j-m+1}}. \end{aligned} \tag{3.4}$$

In order to investigate the rate of convergence of the series in root vectors, let us first consider the series in the eigenvectors of the operator L. We note that

$$c_1 \|f\|_{q+1} \leq \|Lf\|_q \leq c_2 \|f\|_{q+1}, \tag{**}$$

where $c_1 \leq c_2$; and $\|\cdot\|_q$ is the norm in H^q. Such types of estimates are well known in the theory of elliptic operators. *If $Lf \in H^q$, then its series in eigenvectors of L converges in H^q*. Therefore the series for f converges in H^{q+1}. This argument holds for the series in root vectors provided that (**) holds

for A. If A^{-1} exists then ord $A = 1$, ord $A^{-1} = -1$, because $A = L(I + L^{-1}Q)$, $A^{-1} = (I + L^{-1}Q)^{-1} L^{-1}$. Therefore (**) holds for A. We conclude that if $f \in H^q$ then its series in the root vectors of A converges in H^q. This means that the smoother f is, the better its series in the root vectors converges.

One can estimate the remainder of the series. For example, if $h = Lf \in H^0 = L^2(\Gamma)$, then $\sum_{j=N}^{\infty} c_j f_j = \sum_{j=N}^{\infty} (d_j f_j / \lambda_j)$, where $Lf_j = \lambda_j f_j$, $f = \sum_{j=1}^{\infty} c_j f_j$, $h = \sum_{j=1}^{\infty} d_j f_j$. Therefore $|\sum_{j=N}^{\infty} c_j f_j| \leq (1/\lambda_N)(\sum_{j=N}^{\infty} |d_j|^2)^{1/2} \leq |h|/\lambda_N$. It was proved in [31] that the series in eigenvectors of L and root vectors of A are equiconvergent if $p(1-a) > 2$.

Finally let us note that the root vectors are absent if A is normal; that is, $AA^* = A^*A$. This condition can be considered as a condition concerning the surface Γ. It can be writen [25] as

$$\int_\Gamma \frac{\sin(k\,|x-s| - |s-y|)}{|x-s|\,|s-y|}\, ds = 0 \qquad \text{for all } x, y \in \Gamma. \tag{3.5}$$

For the cases when Γ is a sphere or a line the operator A is normal and the EEM method in these cases takes its "engineering" form (without root vectors).

3.2. *Justification of the Asymptotic* SEM

In Section 1 we gave conditions (1.23)–(1.25) sufficient for the validity of the asymptotic SEM defined in (1.26). Condition (1.23) was established in Sections 1 and 2 under weak restrictions which cover the practical cases. We complete the arguments given in Sections 1 and 2 by proving Proposition 1.1. The proof is taken from [23].

Proof of Proposition 1.1. Let $f_1, \ldots, f_n$ be a basis of $N(I + T(z))$, where z is an isolated point where $I + T(z)$ is not invertible. We will show that z is a pole of the operator $(I + T(k))^{-1}$ and its Laurent coefficients are finite-rank operators. Consider the operator $B(k) = I + T(k) + \sum_{j=1}^{n} (\cdot, f_j) g_j$, where $\{g_j\}$, $1 \leq j \leq n$, is a basis of $N(I + T(k)^*)$. Let us show that $N(B(z)) = \{0\}$. Indeed, if $B(k)f = 0$, then

$$(I + T(z))f = -\sum_{j=1}^{n} (f, f_j)\, g_j. \tag{*}$$

Since $g_j \in N(I + T(k)^*)$ they are orthogonal to Ran$(I + T(z))$. (Ran A is the range of the operator A.) Therefore from (*) we conclude that $(f, f_j) = 0$. $(I + T(z))f = 0$. Since $\{f_j\}$ is a basis of $N(I + T(z))$ we get $f = 0$. Therefore $B^{-1}(z)$ is invertible and $B^{-1}(k)$ is invertible if $|k - z| < \delta$, where $\delta > 0$ is some small number. The equation $(I + T(k))\, h = f$ is equivalent to $B(k)h =$

$f + \sum_{j=1}^{n} (h, f_j) g_j$, or to the system $h = B^{-1}(k) f + \sum_{j=1}^{n} c_j B^{-1}(k) g_j$, $c_j = (h, f_j)$. From this it follows that

$$\sum_{j=1}^{n} [\delta_{lj} - b_{lj}(k)] c_j = d_l(k), \qquad 1 \leqslant l \leqslant n, \qquad b_{lj}(k) \equiv (B^{-1}(k) g_j, g_l) \tag{3.7}$$

and $d_l(k) = (B^{-1}(k) f, g_l)$. The functions $b_{lj}(k)$ and $d_l(k)$ are analytic in $|k - z| < \delta$. From Kramer's formulas, it follows that each $c_j(k)$ has a pole at $k = z$ and from (3.6) we can see that the Laurent coefficients of $(I + T(k))^{-1}$ are finite-rank operators.

We now turn to conditions (1.24), (1.25). Unfortunately the known proofs of these conditions given in the papers cited in Section 2.2 are not easy. Therefore we restrict ourselves to a remark concerning condition (1.24). Suppose that (1.24) holds with some real a (even negative). From the Helmholtz equation (1.16) it follows that

$$v = -\frac{f}{k^2} - \frac{\nabla^2 v}{k^2}. \tag{*}$$

Suppose that f is a smooth function which is zero near Γ and near infinity. Then $\nabla^2 v$ satisfies (1.16)–(1.18) with f substituted by $\nabla^2 f$. Therefore $\nabla^2 v$ satisfies inequality (1.24). From this and (*) it follows that v satisfies inequality (1.24) with a substituted by $a + 2$. This argument shows that for functions f that are smooth and compactly supported in Ω we can estimate (1.24) if we only know that $v(x, k)$ does not grow faster than a polynomial as $|\mathrm{Re}\, k| < \infty$, $\mathrm{Im}\, k = \mathrm{const}$. The idea of all the known proofs of (1.25) is to show that the Green's function is small if $|\mathrm{Re}\, k| \to \infty$ and $|\mathrm{Im}\, k| < \phi(|\mathrm{Re}\, k|)$, where $\phi(r) > 0$ is a nondecreasing function $\phi(r) \to +\infty$ as $r \to \infty$. For the three-dimensional potential scattering it was proved in [22] that $\phi(r) = a + b \ln r$, $b > 0$. For diffraction problems in case of a smooth scatterer (Dirichlet or Neumann boundary conditions) $\phi(r) \sim r^{1/3}$ as $r \to \infty$, while for a nonsmooth scatterer $\phi(r) \sim \ln r$ as $r \to \infty$ (see (2.16) and (2.17)).

3.3. *A Variational Principle for the Spectrum of Compact Nonselfadjoint Operators*

Let T be a compact linear operator on a Hilbert space H with eigenvalues λ_j, $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots$. Let $r_j = |\mathrm{Re}\, \lambda_{m(j)}|$ be ordered so that $r_1 \geqslant r_2 \geqslant \cdots$ and $t_j = |\mathrm{Im}\, \lambda_{n(j)}|$ be ordered so that $t_1 \geqslant t_2 \geqslant \cdots$. The indices $m(j)$ and $n(j)$ make the ordering. Let L_j be the eigenspace of T corresponding to λ_j, and $M_j(N_j)$ be the eigenspace corresponding to $r_j(t_j)$ (that is, to $\lambda_{m(j)}(\lambda_{n(j)})$). Let $\tilde{E}_j = \sum_{m=1}^{j} \dotplus L_m$, and $\tilde{M}_j, \tilde{N}_j$ are defined similarly. The sign $\dotplus$ denotes the direct sum. Let $\perp\!\!\!\perp$ denote the direct complement in H.

PROPOSITION 3.2. *The following formulas hold:*

$$|\lambda_j| = \max_{x \in \tilde{L}_{j-1}^{\perp\!\!\perp}} \min_{\substack{y \in H \\ (x,y)=1}} |(Tx, y)|,$$

$$r_j = \max_{x \in \tilde{M}_{j-1}^{\perp\!\!\perp}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Re}(Tx, y)|,$$

$$t_j = \max_{x \in \tilde{N}_{j-1}^{\perp\!\!\perp}} \min_{\substack{y \in H \\ (x,y)=1}} |\mathrm{Im}(Tx, y)|.$$

A proof is given in [31]. It would be interesting to try this variational principle numerically.

3.4. *Variational Principle and Perturbation Theory for Resonances*

In this section we prove existence of the limit (2.9) and orthogonality of the resonant states corresponding to different k_1, k_2, $\mathrm{Im}\, k_1 < 0$, $\mathrm{Im}\, k_2 < 0$ with respect to the form $\langle \cdot, \cdot \rangle$ defined in (2.3). Our argument is close to the one in [30]. In order to prove existence of the limit (2.9), it is sufficient to prove existence of the limit

$$\langle u, v \rangle = \lim_{\varepsilon \to +0} \int_{|x| > R} u(x)\, v(x) \exp(-\varepsilon r \ln r)\, dx, \qquad r = |x|. \tag{3.8}$$

For $|x| \geqslant R$ the functions u, v can be represented by the series (2.8). These series converge uniformly in $n \in S^2$ (S^2 is the unit sphere in $\mathbb{R}^3$) and absolutely. Therefore it is sufficient to prove the existence of the limits

$$\lim_{\varepsilon \to +0} \int_R^\infty \exp(-\varepsilon r \ln r)\, r^{-j} \cdot \exp(br + iar)\, dr,$$

where $b = -\mathrm{Im}(k_1 + k_2) > 0$, $a = \mathrm{Re}(k_1 + k_2) \neq 0$, $j \geqslant -2$. Suppose that $a > 0$. Let $0 < \theta < \pi/2$, $C_N = \{z : |z - R| = N,\ 0 \leqslant \arg(z - R) \leqslant \theta\}$, $C_{\theta N} = \{z : \arg(z - R) = \theta,\ 0 \leqslant |z - R| \leqslant N\}$, $C_R = \{z : R \leqslant z \leqslant R + N\}$, $C_\theta = C_{\theta\infty}$, $C = C_N \cup C_{\theta N} \cup C_R$. We have $\int_{C_N} \exp(-\varepsilon r \ln r)\, r^{-j} \exp(br + iar)\, dt \to 0$ as $N \to \infty$, $\varepsilon > 0$. Therefore

$$\int_R^\infty \exp(-\varepsilon r \ln r)\, r^{-j} \exp(br + iar)\, dr = \int_{C_\theta} \exp(-\varepsilon r \ln r)\, r^{-j} \exp(br + iar)\, dr. \tag{3.9}$$

Let us choose $0 < \theta < \pi/2$, such that $\sin \theta > (b/a) \cos \theta$. Then the integral (3.9) will be absolutely convergent for $\varepsilon = 0$ and (3.8) is proved. The case $a < 0$ is treated similarly with θ substituted by $-\theta$. It is easy to prove the

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orthogonality of the resonant states, corresponding to $k_1^2 = k_2^2$, with respect to the form (3.8). Indeed, let us multiply the identity $v(\nabla^2 + k_1^2)u - u(\nabla^2 + k_2^2)v = 0$ in Ω by $\exp(-\varepsilon r \ln r) \equiv f(r, \varepsilon)$, integrate over $\Omega_R = \{x: |x| \leqslant R, x \in \Omega\}$, and take first $R \to +\infty$ and then $\varepsilon \to +0$. Then use the Green's formula. The terms which appear because of the differentiation of $f(r, \varepsilon)$ when we integrate by parts will tend to 0 as $\varepsilon \to 0$. As a result we get $\langle u, v\rangle = 0$. This is what we wanted to show.

Remark. For $a = \mathrm{Re}(k_1 + k_2) = 0$ our argument is not valid.

We now turn to the proof of the statement of Section 2.4. We assume that the operator $I + T(z)$ is not invertible. Let $\phi_1, ..., \phi_n$ be an orthonormal basis of $N(I + T(z))$, that is, let $(\phi_i, \phi_j) = \delta_{ij}$, $\psi_1, ..., \psi_n$ be an orthonormal basis of $N(I + T(z)^*)$. Let $Qh = \sum_{j=1}^n (h, \phi_j)\psi_j$. First let us show that the operator $I + T(z) + Q$ is invertible in H. Since $T(z) + Q$ is compact we only need to prove that $N(I + T(z) + Q) = \{0\}$. Suppose that $(I + T(z))h = -\sum_{j=1}^n (h, \phi_j)\psi_j = -Qh$. Then by the Fredholm alternative we conclude $(Qh, \psi_i) = 0$, $1 \leqslant i \leqslant n$. Thus $(h, \phi_i) = 0$, $1 \leqslant i \leqslant n$, $(I + T(z))h = 0$. Therefore $h = 0$. We have proved that $\Gamma = (I + T(z) + Q)^{-1}$ exists. In order to study $(I + T(k, \varepsilon))^{-1}$ let us write

$$(I + T(k, \varepsilon))^{-1} = (I - a(\lambda, \varepsilon))^{-1}\,\Gamma(\lambda, \varepsilon),$$

where $\Gamma(\lambda, \varepsilon)$ is analytic in $\lambda = k - z$ and ε, $\Gamma(0, 0) = \Gamma$, and $a(\lambda, \varepsilon) = (I + T(z) + Q + T(k, \varepsilon) - T(z))^{-1} Q$. Since $a(\lambda, \varepsilon)$ is a finite-rank operator (because Q is) we can use a matrix representation of $a(\lambda, \varepsilon)$ and write

$$(I - a(\lambda, \varepsilon))^{-1} = \frac{A(\lambda, \varepsilon)}{\Delta(\lambda, \varepsilon)}.$$

Here $\Delta(\lambda, \varepsilon) = \det(\delta_{ij} - a_{ij}(\lambda, \varepsilon))$, $A = (A_{ji}(\lambda, \varepsilon))$ is the algebraic complement to $\delta_{ji} - a_{ji}(\lambda, \varepsilon)$, $1 \leqslant i, j \leqslant n$. By our assumption the operator $A(\lambda, 0)/\Delta(\lambda, 0)$ has a pole $\lambda = 0$. Let m be its order. This means that

$$\frac{A(\lambda, 0)}{\Delta(\lambda, 0)} = \frac{A_0 + A_1 \lambda + O(\lambda^2)}{\lambda^m(\Delta_0 + \lambda \Delta_1 + O(\lambda^2))}, \qquad \Delta_0 \neq 0, \quad A_0 \neq 0.$$

We have

$$\begin{aligned} A(\lambda, \varepsilon) &= A(\lambda, 0) + \varepsilon A_1(\lambda, 0) + O(\varepsilon^2) \\ &= A_0 + \lambda A_1 + O(\lambda^2) + \varepsilon A_1(0, 0) + \cdots. \end{aligned}$$

To the function $\Delta(\lambda, \varepsilon)$ we apply the Weierstrass preparation theorem [8]. The statement of this theorem is given below for the convenience of the reader.

THEOREM (Weierstrass' preparation theorem). *Let $F(\lambda, \varepsilon)$ be holomorphic in a neighborhood of $(0, 0)$, $F(\lambda, 0) = \lambda^m f(\lambda)$, $f(0) \neq 0$. Then there exists a holomorphic function $g(\lambda, \varepsilon)$, $g(0, 0) \neq 0$, and holomorphic functions $A_j(\varepsilon)$ such that*

$$F(\lambda, \varepsilon) = \left[\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j\right] g(\lambda, \varepsilon),$$

$$A_j(0) = 0, \quad 1 \leqslant j \leqslant m - 1.$$

From this theorem it follows that $\Delta(\lambda, \varepsilon) = [\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j]\, g(\lambda, \varepsilon)$, $A_j(0) = 0$. It is now clear that the singularities of $(I - a(\lambda, \varepsilon))^{-1}$ are determined by the function $[\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j]^{-1}$. The equation $\lambda^m + \sum_{j=0}^{m-1} A_j(\varepsilon)\lambda^j = 0$ has $p \leqslant m$ different roots $\lambda_j(\varepsilon)$, $\lambda_j(0) = 0$. These roots can be represented by the series in powers of $\varepsilon^{1/r}$, where $r > 0$ is some integer. There is an algorithm (method of the Newton diagram) in the literature for construction of these series (Puiseux series) [13]. But our argument has already proved the statement of Section 2.4.

4. Examples, Comments, and Some Additional Material

4.1. *Examples*

1. Consider the matrix $T(k) = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$. This is an analytic operator function on $H = \mathbb{R}^2$. Its resolvent is $(T(k) - \lambda I)^{-1} = \begin{pmatrix} 1-\lambda & -k \\ -k & 1-\lambda \end{pmatrix} / (\lambda^2 - 1 - k^2)$. Its eigenvalues are $\lambda_\pm = \pm\sqrt{1 + k^2}$. *For any fixed λ, the resolvent is a meromorphic in k function (as it should be according to Proposition 1.1), but the eigenvalues as functions of k have branch points.*

2. *A symmetric (with respect to the form $[f, g] = \int f(x)\, g(x)\, dx$) nonselfadjoint operator can have root vectors.*

EXAMPLE. $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is an operator on $\mathbb{C}^2$ symmetric with respect to the form $[x, y] = x_1 y_1 + x_2 y_2$, that is $[Ax, y] = [x, Ay]$. The operator $(A - \lambda I)^{-1}$ has a pole of order 2 at $\lambda = 0$. The corresponding eigenvector is $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and root vector is $\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$.

3. *The fact that the algebraic problem to which an original integral equation was reduced (e.g., by a projection method, in particular by the method of moments) has eigenvalues does not guarantee that the original equation has eigenvalues.* For example, $Vf = \int_0^x f(t)\, dt$ has no eigenvalues, but any $n \times n$ matrix has eigenvalues. Proposition 2.2 says that if the original equation has eigenvalues these eigenvalues can be calculated by the projection method described in Section 2. On the other hand if n is a number of the basis functions used in the projection method and $\lambda_j^{(n)}$ is the jth eigen-

value of the operator $T_n = P_n T P_n$, where P_n is the projection on the n-dimensional space spanned by the basis functions, then the limit point λ_j as $n \to \infty$ of the sequence $\lambda_j^{(n)}$ is an eigenvalue of T (under weak assumptions about T and the basis functions; e.g., if T is compact and the basis functions form an orthonormal set).

4. *There exists an analytic (in k) compact operator $T(k)$ such that $(I+T(k))^{-1}$ has multiple poles but $T(k)$ is diagonalizable for all k, that is, for any k the operator $T(k)$ has no root vectors. This means that although the* EEM *(as defined in Section* 1) *can be applied in the form* (1.10), *the operator $(I+T(k))^{-1}$ has multiple poles.*

EXAMPLE.

$$T(k) = \begin{pmatrix} -1+k^2 & 0 \\ 0 & k^2 \end{pmatrix}, \qquad (I+T(k))^{-1} = \begin{pmatrix} 1/k^2 & 0 \\ 0 & 1/1+k^2 \end{pmatrix},$$

anf for any k, $T(k)$ is diagonal and therefore has no root vectors.

5. *In the finite-dimensional space $\mathbb{R}^n$ every linear operator which has n linearly independent eigenvectors is similar to a normal operator; it is diagonal in its eigenbasis (that is in the basis consisting of its eigenvectors). In the Hilbert space there exists an operator with eigenvectors which span H but which is not similar to a normal operator.*

An example can be found in [6]. Since this example is rather technical we will not give it here. It seems to be of no practical use for engineers.

6. *Whether a root system forms a basis of H or not can depend on the choice of the root system if the total number of the root vectors is infinite.*

EXAMPLE. Let $-y'' = \lambda y$, $0 \leqslant x \leqslant 1$, $y(0) = 0$, $y'(0) = y'(1)$; $H = L^2([0,1])$. The eigenvalues of this problem are $\lambda_n = (2\pi n)^2$, $n = 0, 1, 2$ and the corresponding eigenvectors and the root vectors are $y_0 = x$, $y_n = \sin 2\pi n x$, and $y_n^{(1)} = x\cos(2\pi n x)/4\pi n$. One can easily check that the biorthogonal system to the above root system is $v0 = 2$, $v_n = 4(1-x)\sin(2\pi n x)$, and $v_n^{(1)} = 16\pi n \cos(2\pi n x)$. Consider now a different choice of the root vectors. Let $z_n^{(1)} = y_n^{(1)} + y_n$. The system biorthogonal to $\{y_n, z_n^{(1)}\}$ is $\{v_n - v_n^{(1)}, v_n^{(1)}\}$. Thus $\|z_n^{(1)}\| \, \|v_n^{(1)} - v_n\| = (1 + O(1/n^2)) \cdot O(n^2) \to \infty$ as $n \to \infty$. *This means that the system $\{y_n, z_n^{(1)}\}$ is not a basis of H, because in order for a complete minimal system $\{\phi_n\}$ to be a basis it is necessary that $\sup_n \|\phi_n\| \|\psi_n\| \leqslant c$, where $\{\psi_n\}$ is the system biorthogonal to $\{\phi_n\}$*. Let us explain the last statement. If $\{\phi_n\}$ is a basis and $\{\psi_n\}$ is the biorthogonal system, that is, $(\psi_j, \phi_n) = \delta_{jn}$, then the expansion of an arbitrary element $f \in H$ takes the form $f = \sum_{j=1}^{\infty} c_j \phi_j$ with $c_j = (\psi_j, f)$ The norms of the operators S_n, $S_n f = \sum_{j=1}^{n} (\psi_j, f)\, \phi_j$ are bounded uniformly in n because $\|S_n f - f\|_{n\to\infty} \to 0$ for any $f \in H$, where $\|\cdot\|$ is the norm of an element of H. Therefore the norm of the operator $S_n - S_{n-1} = (\psi_n, \cdot)\, \phi_n$ is bounded uniformly in n. But $|(\psi_n, \cdot)\, \phi_n| = \|\phi_n\| \, \|\psi_n\|$, where $|\cdot|$ is the norm of an operator on H. *We proved that the condition*

$$\sup_n \|\phi_n\| \, \|\psi_n\| \leqslant c \tag{4.1}$$

is necessary for a complete minimal system to form a basis of H. In (4.1), $\{\psi_n\}$ is the biorthogonal to $\{\phi_n\}$ system. It is known that there exists a unique system biorthogonal to a complete minimal system. The system we gave in the example was used in [9].

4.2. *Target Identification*

An interesting problem both theoretically and practically is the inverse problem of identification of the obstacle (target) from the set of complex poles of the Green's function corresponding to this target. No solution to this problem is known. The author thinks that in order to use the complex poles for target identifacition it is more useful from the practical point of view to have tables of the poles for some typical scatterers (say, aircrafts of various kinds) rather than to use some theoretical results. These few results will be mentioned below. At present time there is an experimental technique which gives a possibility of finding several complex poles corresponding to a given scatterer. It is an interesting theoretical problem to develop an optimization-type numerical technique in order to calculate the poles from the experimental data (see Section 5). It was observed in [14] that for a star-shaped obstacle which contains a ball of radius R_1 and is confined in the ball of radius R_2 the number $N(\tau)$ of the purely imaginary complex poles $-i\tau_n$, $0 < \tau_n < \tau$, satisfies the inequalities

$$CR_1^2 \leqslant \limsup_{\tau\to\infty} \frac{N(\tau)}{\tau^2}, \qquad \limsup_{\tau\to\infty} \frac{N(\tau)}{\tau^2} \leqslant CR_2^2, \tag{4.2}$$

where $C = 1.138370$. Theoretically this gives some information about the scatterer if the asymptotics of large purely imaginary poles is available. But practically one can find from the experimental data only several poles ordered according to the growth of $|\mathrm{Im}\, k_j|$ (poles nearest to the real axis). These poles in general are not purely imaginary. Therefore from the practical point of view it is difficult to make use of (4.2). Let us mention some related results. For interior problem the set of all eigenvalues (which are the poles of the Green's function of the interior problem) does not define the shape of the body uniquely.

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For potential scattering on the semiaxis, the set of the poles of the Green's function does not define the potential uniquely. There exists an r-parametric family of potentials having the same set of poles of the Green's functions. Here r is the number of the bound states, that is, complex poles with positive imaginary parts. Since this observation seems to be new we will give some details. From the theory of the potential scattering for central potentials it is known [17, Chap. 12] that the Jost function can be represented in the form

$$f(k)=f(0)\exp(ikR)\prod_{n=1}^{\infty}\left(1-\frac{k}{k_n}\right), \tag{4.3}$$

where we assume (without loss of generality) that $f(0)\neq 0$. In (4.3) the numbers k_n are the poles of the Green's function of the Schrödinger operator $ly=-y''+V(r)y$, $y(0)=0$, $0<r<\infty$. There can be equal poles in (4.3). We assume that $V(r)=0$ for $r>R$. The Jost function $f(k)=f(0,k)$, where $f(r,k)$ is the solution of the problem $ly-k^2y=0$, $r>0$, $y=\exp(ikr)+o(1)$ as $r\to\infty$. Thus if we know the poles of the Green's function we can find $f(k)$ by formula (4.3). If we know $f(k)$ we know the phase shift and the bound states. The phase shift $\delta(k)$ is to be found from the formula $\exp(2i\delta)=f(-k)/f(k)=S(k)$, where $S(k)$ is the S-matrix [17]. These data and r arbitrary positive parameters (the normalization constants) are sufficient for constructing the potential $V(r)$ which has the above scattering data. The algorithm for the reconstruction of $V(r)$ is the well-known inverse scattering theory [5]. In particular, the potential $V(r)$ can be uniquely determined from the knowledge of the complex poles iff the imaginary part of each pole is negative.

4.3. *Infiniteness of the Number of Complex Poles*

From (4.2), it follows that if the scatterer is star-shaped, then its Green's function has infinitely many purely imaginary poles. It is not proved that there are infinitely many complex poles k_j with $\operatorname{Re} k_j\neq 0$. Heuristic arguments (e.g., formulas (2.15) and (2.18)) show that there are infinitely many such poles. It would be interesting to prove it. For three-dimensional scattering for a noncentral potential) this problem is open also.

For potential scattering on the semiaxis it is proved that there are infinitely many complex poles k_j with $\operatorname{Re} k_j\neq 0$ [17]. Let us give another proof that *there are infinitely many purely imaginary complex poles of the Green's function of the exterior Dirichlet Laplacian.* A proof of this statement was given in [14]. Our proof is different, but we use an idea from [14]. Our starting point is Proposition 2.3.

Let $k=-ib$, $b>0$, be a complex pole. Then the equation $A(b)f\equiv\int_\Gamma G_0(s,s',-ib)f(s')\,ds'=0$ has a nontrivial solution, $G_0(s,s',-ib)=\exp(b|s-s'|)/4\pi|s-s'|$. The operator $A(b)$ is selfadjoint in $H=L^2(\Gamma)$ if $b>0$ and analytic in b. Therefore [12, Chap. 2, Sect. 6] its eigenvalues $\lambda_n(b)$ are analytic in b in a neighborhood of the real axis of the complex plane b. If $b\leqslant 0$ the operator $A(b)>0$ in H and $\lambda_n(b)>0$. When $b>0$ and $b\to+\infty$ the number N_- of the negative eigenvalues of $A(b)$ goes to infinity. Since $\lambda_n(b)$ are analytic in b (continuously would be sufficient for our purpose) they vanish at some point b_n before they become negative. This point b_n is a complex pole according to Proposition 2.3. Therefore, if we prove that

$$N_-\to\infty \qquad \text{as} \quad b\to\infty \tag{*}$$

we prove that there are infinitely many complex poles $-ib_n$. Let us prove (*). Let us take a point inside Γ and draw some lines $l_1,\ldots,l_n$ intersecting at this point. Let s_n, s_n' be the points of the intersections of l_n with Γ. Let us choose a function h_n which is equal to 1 (-1) in a small neighborhood $S_n(S_n')$ of $s_n(s_n')$ and vanishes outside of these neighborhoods. We assume that $S_n\cap S_m=\varnothing$, $n\neq m$, $S_n\cap S_n'=\varnothing$. In this case the system $\{h_1,\ldots,h_n\}$ is linearly independent. If $(A(b)h,h)<0$ for $h\in L_n$ and b is sufficiently large, then $A(b)$ has at least n negative eigenvalues. Here L_n is the linear span of $\{h_1,\ldots,h_n\}$. If $h=\sum_{j=1}^n c_jh_j$, then $(A(b)h,h)=\sum_{i,j=1}^n a_{ij}c_ic_j$, where

$$\begin{aligned} a_{ij} &= \int_{S_i\cup S_i'}\int_{S_j\cup S_j'} \frac{\exp(b|x-y|)}{4\pi|x-y|}\,h_i(y)\,h_j(x)\,dy\,dx \\ &= \frac{a^4}{4\pi}\left(\frac{\exp(b|s_i-s_j|)}{|s_i-s_j|}+\frac{\exp(b|s_i'-s_j'|)}{|s_i'-s_j'|}\right. \\ &\quad\left.-\frac{\exp(b|s_i-s_j'|)}{|s_i-s_j'|}-\frac{\exp(b|s_i'-s_j|)}{|s_i'-s_j|}\right), \end{aligned}$$

where a^2 is the area of S_n, S_n' and

$$i\neq j, \qquad a_{ij}\approx-\frac{2\exp(b|s_j-s_j'|)}{4\pi|s_j-s_j'|}\,a^4+O\left(\frac{2\exp(ba)}{4\pi a}\right).$$

We can choose lines l_j, $1\leqslant j\leqslant n$ so that $\max_{i\neq j}|s_i-s_j|<\min_j|s_j-s_j'|$. In this case for $b>0$ sufficiently large the matrix a_{ij} will be negatively definite, because the diagonal elements $a_{jj}<0$, $1\leqslant j\leqslant n$, and dominate if b is sufficiently large. This completes the proof. We make no assumptions about convexity or even star-shapedness of Γ.

Remark. Suppose that Γ_1 and Γ_2 are homothetic and q is the homothety coefficient, that is, $\Gamma_2=q\Gamma_1$, $q>1$. Then $b_j^{(1)}=qb_j^{(2)}$, where $-ib_j^{(1)}$ and $-ib_j^{(2)}$, $1\leqslant j\leqslant\infty$ are the purely imaginary poles of the Green's function of

the Dirichlet Laplacian in the exterior of Γ_1 and Γ_2, respectively. This can be verified by changing variables ($y_2 = qy_1$, $x_2 = qx_1$) in the equation

$$\int_{\Gamma_1} \frac{\exp(b_j^{(1)}|x_1 - y_1|)}{|x_1 - y_1|} f(y_1)\, dy_1 = 0$$

corresponding to the pole $-ib_j^{(1)}$. From the results in [14] it follows that if D_1 and D_2 are star shaped, then $N_1(b) \leqslant N_2(b) \leqslant N_3(b)$, where $N_j(b)$ is the number of purely imaginary poles of the Green's function of the Dirichlet Laplacian in the exterior to D domain modulo $\leqslant y$. Since for a ball $N(b) \sim cR^2b^2$, $c = 1.13837$, R is the radius of the ball, $b \gg 1$, this gives another proof that $N(b) \to \infty$ as $b \to +\infty$ for any D.

4.4. *Behavior of Solutions to the Wave Equation as $t \to +\infty$*

In SEM the information about the behavior of solutions to the wave equation as $t \to +\infty$ is obtained (see (1.26)) because some analytic properties of the solution to the corresponding stationary problems are known ((1.23))–(1.25)). In this section we will point out a general result which says that for a wide class of abstract operators (when analytic continuation of the resolvent kernels of the operators is not necessarily possible) there is a one-to-one correspondence between asymptotic behavior of solution to the abstract wave equation in a Hilbert space

$$u_{tt} + Lu = f\exp(i\omega t), \qquad u(0) = u_t(0) = 0, \tag{4.4}$$

when $t \to +\infty$ and analytic properties of the resolvent of L in a neighborhood of the spectrum of L. Since these results [27] are of a mathematical nature and their statement is not sufficiently short for including it in this paper, we want only to mention some points of possible interest in applications. First, all the problems studied in the applications can be formulated as (4.4) with L satisfying conditions from [27]. Second, it is proved in [27] that the limiting amplitude principle is equivalent to the limiting absorption principle. The limiting amplitude principle says that there exists $\lim_{T\to\infty}(1/T)\int_0^T \exp(-i\omega t)\, Pu(t)\, dt = Pv$, where in applications P is the orthogonal projection on $L^2(\tilde{\Omega})$, where $\tilde{\Omega}$ is a compact subdomain of Ω, and v is the solution to the stationary problem $(L - k^2)v = f$. The limiting absorption principle says that the following limit exists:

$$\lim_{\varepsilon\to+0} Pv(k + i\varepsilon) = Pv(k), \qquad v(k + i\varepsilon, f) \equiv [L + (k + i\varepsilon)^2 I]^{-1}.$$

Third, some formula of Tauberian type was proved in [27] but without usual Tauberian conditions (of the type $u(t) \geqslant 0$ or $u(t) > c$) which are very difficult to verify in practical problems (and theoretical problems in partial differential equations as well). This formula gives a relation between the

asymptotic behavior of a function as $t \to +\infty$ and asymptotic behavior of its Laplace transform as $p \to 0$.

5. Problems

(1) Is it true that $A(k)$, $T(k) \in R(H)$? In Section 3.1 we proved that $A(k)$, $T(k) \in R_b(H)$. The question is: Does basisness without brackets hold?

(2) What is the relation between the order of a complex pole and the multiplicity of the zeros of $\lambda_n(k)$? (See proposition 2.3).

(3) Can the scatterer be uniquely identified by the set of complex poles of the corresponding Green's function?

(4) Prove that there are infinitely many complex poles k_j with $\mathrm{Re}\, k_j \neq 0$ (in diffraction problems and noncentral potential scattering).

(5) Are the complex poles of the Green's function of the exterior Dirichlet or Neumann Laplacian simple?

(6) Make numerical experiments in the calculation of the complex poles.

(7) Prove convergence of the numerical procedure for calculation of the complex poles suggested in [30].

(8) Find a theoretical approach optimal in some sense to approximate a function $f(t)$ by the functions of the form $f_N = \sum_{j=1}^N \sum_{m=1}^{m_j} \exp(-ik_jt)\, t^{m-1} c_{mj}$. Here the number c_{mj}, m_j, k_j are to be found so that f_N will approximate $f(t)$ in some optimal way. Currently some methods (e.g., Proney method) are used in practice, but they are not optimal. This problem seems to be of general interest (optimal harmonic analysis in complex domain).

(9) When can SEM in the form of (1.28) be justified?

6. Conclusion

We hope to have shown in this paper that:

(1) EEM is justified (in the generalized form of expansion in root vectors).

(2) SEM is justified in the asymptotic form (1.26).

(3) Numerical projection method for calculation of the complex poles is justified.

(4) There are many interesting and difficult open problems in the field.

(5) Numerical results and experiments are desirable.

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Convergence of the T-matrix approach to scattering theory[a)]

APPENDIX IV

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(Received 22 July 1981; accepted for publication 13 November 1981)

The T-matrix numerical scheme is widely used in practice. Convergence of this scheme was not proved. A proof of convergence is given in this paper.

PACS numbers: 03.80. + r, 11.20. − e, 02.30.Jr

1. INTRODUCTION

At the international symposium on wave scattering[1] most of the speakers pointed out that the T-matrix scheme needed a justification, its convergence was not proved. In this paper a proof of convergence is given. This proof also clarifies another basic question, namely, convergence of the variational method of finding stationary points of functionals.[6] Many physical problems are formulated as the problems of finding stationary points and/or stationary values of some functionals, and these points are not extremal. A necessary and sufficient condition for a stationary principle to be extremal is given in Ref. 2, p. 90. The standard T-matrix approach is described in Ref. 1, pp. 64. The principal difference between the standard and our approach is as follows. In the standard approach the scattered field is represented as the series in the outgoing spherical waves and the coefficients of the series are found from a linear system. One assumes that the series converges on Γ (the Rayleigh hypothesis) which is not true in general. In our approach one uses a basis of $L^2(\Gamma)$ and no difficulties with convergence arise.

Let us describe a modified T-matrix approach to the problem

$$(-\nabla^2 - k^2)u = f \quad \text{in } \Omega, \quad k > 0 \tag{1}$$

$$u|_\Gamma = 0, \quad r(\partial u/\partial r - iku) \to 0 \quad \text{as } r = |x| \to \infty \tag{2}$$

Here Ω is an exterior domain with a smooth closed boundary Γ and $D = \mathbb{R}^3 \setminus \Omega$ is a bounded domain. From the Green formula it follows that

$$u(x) = v(x) - \int_\Gamma g(x,s)h(s)ds, \quad x \in \Omega \tag{3}$$

$$0 = v(x) - \int_\Gamma g(x,s)h(s)ds, \quad x \in D \tag{4}$$

$$v(x) = \int_\Omega g f\, dy, \quad g = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \quad h = \frac{\partial u}{\partial N}. \tag{5}$$

N is the unit normal to Γ directed into Ω. If h is found, then $u(x)$ can be found from (3). Let us rewrite Eq. (4) as

$$Ah = v(s), \quad Ah = \int_\Gamma g(s,s')h\,ds', \quad s \in \Gamma. \tag{6}$$

Let $\{\phi_j\}, j = 1,2,\ldots$ be a basis of $H_{-1/2}$, where $H = H_0 = L^2(\Gamma)$, $H_q = W_2^q(\Gamma)$ are the Sobolev spaces,[14]

$$h_n = \sum_{j=1}^n c_j \phi_j, \tag{7}$$

$$\sum_{m=1}^n a_{jm} c_m = v_j, \tag{8}$$

where

$$a_{jm} = (A\phi_m, \phi_j), \quad v_j = (v,\phi_j), \quad (f,v) = (f,v)_H. \tag{9}$$

Let (t_{jm}) be the inverse matrix $a_{jm}^{(-1)}$, $1 \le j,m \le n$. Then c_j, $1 \le j \le n$, can be calculated if v_j, $1 \le j \le n$, are given. From formula (7) one calculates h_n, and from formula (3) with $h = h_n$ one calculates the approximate solution u_n to problem (1)–(2). The problem is to prove that (i) for sufficiently large n the matrix a_{jm} in (8) is invertible, (ii) $\|h_n - h\| \to 0$ as $n \to \infty$, where $\|\cdot\|$ is the norm in H. Actually, convergence will be proved in H_q, where q depends on f and on the smoothness of Γ. Let us assume for simplicity that $\Gamma \subset C^\infty$. Then q depends on the smoothness of f if $\text{meas}(\Gamma \cap \text{supp} f) > 0$ ($\text{supp} f$ is the support of f) and q is arbitrary if $\text{dist}(\text{supp} f, \Gamma) > 0$.

The basic idea of the proof is very simple and is given in Sec. 2. In Sec. 3 some technical details are provided.

2. MAIN RESULT

Theorem 1: System (8) is uniquely solvable for sufficiently large n and $\|h_n - h\| \to 0$ as $n \to \infty$ [without loss of generality we assume that the operator $I + T(k)$ is invertible; see the proof below, n.1 of Sec. 3].

Proof: The basic idea is to factorize A in (6) as $A = A_0[I + T(k)]$, where $A_0 = A(0)$, $T(k) = A_0^{-1} \times [A(k) - A(0)]$. The operator $A_0 > 0$ is a bijection of H_q onto H_{q+1}, while $T(k)$ is compact in H_q for any q (see, e.g., Ref. 2, p. 287). The system (8) can be written as $(A_0[I+T]h_n, \phi_j) = (v,\phi_j)$, $1 \le j \le n$. Since $A_0 > 0$ the form $(A_0 u, f)$ is a scalar product which we denote by $[u,f] = (A_0 u, f)$. This scalar product generates a norm $[u,u]^{1/2} = \|A_0^{1/2} u\|$ which is equivalent to the norm in $H_{-1/2}$. This follows from the fact that A_0 is a pseudodifferential elliptic operator of order -1 and therefore $\text{ord} A_0^{1/2} = -\frac{1}{2}$. Thus (8) is of the form $[(I+T)h_n, \phi_j] = (v,\phi_j)$, $1 \le j \le n$. Let $w = A_0^{-1} v$. Then $(v,\phi_j) = [w,\phi_j]$ and $h_n + P_n T h_n = P_n w$, where P_n is the orthoprojection in $H_{-1/2}$ on the linear span of $\{\phi_1,\ldots,\phi_n\}$. The operator $I + T(k)$, $k > 0$, can be assumed invertible (this will be shown in Sec. 3), and $T(k)$ is compact on H_q with arbitrary q, $-\infty < q < \infty$. Therefore $\|(I - P_n)T\|_q \to 0$ as $n \to \infty$, and the norm is the norm of operators on H_q (this will be explained in Sec. 3). Thus $I + P_n T = I + T - (I - P_n)T$ is invertible for sufficiently large n. This means that system (8) is uniquely solvable for sufficiently large n. Furthermore,

$$h - h_n = (I+T)^{-1}w - (I + P_n T)P_n w$$
$$= B[I - (I - P^{(n)}TB)^{-1}P_n]w, \tag{10}$$

[a)]Supported by AFOSR 800204.

where $B = (I + T)^{-1}$, $P^{(n)} = I - P_n$. Thus

$$\|h - h_n\|_q < c\|(I - Q_n)^{-1} Q_n P_n w\|_q$$

$$< c_1 \|Q_n\|_q \|P_n w\|_q, \quad c, c_1 = \text{const} > 0, \tag{11}$$

where $Q_n = P^{(n)} TB$, $\|Q_n\|_q \to 0$ as $n \to \infty$. If $\|w\|_q = \|A_0^{-1} v\|_q < \infty$ then (11) shows that $\|h_n - h\|_q \to 0$ as $n \to \infty$ and the rate of convergence is given by the rate of decay of the magnitude $\|P^{(n)} T\|_q$ as $n \to \infty$. In order that $\|A_0^{-1/2} v\|_q < \infty$ it is necessary and sufficient that $v \in H_{q+1/2}$. This is so if $f \in H_{q-1}(\Omega)$ because in this case $v \in H_{q+1}(\Omega)$ and its trace $v|_\Gamma \in H_{q+1/2}$. Our argument shows that if $f \in L^2(\Omega) = H(\Omega)$ the smoothness of $v|_\Gamma$ is even higher than we need. This completes the proof. Theory of the H_q spaces and the trace theorems can be found in Ref. 4.

3. ADDITIONAL DETAILS

(1) Let us show first that $I + T(k)$, $k > 0$, is invertible. Since T is compact, it is sufficient to show that the nullity of this operator is trivial. If $[I + T(k)]h = 0$ then $A(k)h = A_0[I + T(k)]h = 0$. Therefore the function $u(x) = \int_\Gamma g(x, s) h ds$ solves the homogeneous problem (1)–(2). It is well known that the solution of (1)–(2) is unique. Thus $u(x) \equiv 0$ in Ω. If k^2 is not an eigenvalue of the Dirichlet Laplacian in D then $u(x) \equiv 0$ in D, and from the jump relation for the normal derivative of u one derives that $h = 0$. If k^2 is the eigenvalue of the Dirichlet Laplacian in D, then the argument is the same but instead of $g(x, y, k)$ in (3)–(5) one should use the Green function $g_\epsilon(x, y, k)$ of the exterior of a small ball $B_\epsilon \subset D$. This ball is so chosen that k^2 is not an eigenvalue of the Dirichlet Laplacian in $D_\epsilon = D \setminus B_\epsilon$. Obviously such a ball can be found (there are infinitely many such balls).

Remark 1: The idea of applying $g_\epsilon(x, y, k)$ in order to deal with the case when k^2 is an eigenvalue of the interior problem was used in Ref. 3.

(2) Let us show that $\| P^{(n)} T \| \to 0$ as $n \to \infty$. Since $\{\phi_j\}$ is a basis, one has $\| P^{(n)} f \| \to 0$ as $n \to \infty$ for any $f \in H$. Since T is compact it can be written as $T = T_N + d_N$, where $\| d_N \| < \epsilon_N$, $\epsilon_N \to 0$ as $N \to \infty$, and T_N is finite dimensional: $T_N f = \sum_{j=1}^N (f, \psi_j) w_j$. Clearly it is sufficient to prove that $\| P^{(n)} T_N f \| < \delta_n \| f \|$, where $\delta_n \to 0$ as $n \to \infty$. One has $\| P^{(n)} T_N f \| < \sum_{j=1}^N \| P^{(n)} w_j (f, \psi_j) \| < \| f \| \sum_{j=1}^N \| \psi_j \| \| P^{(n)} w_j \| < \delta_n \| f \|$, where $\delta_n \to 0$ as $n \to \infty$ because $\| P^{(n)} w_j \| \to 0$ as $n \to \infty$, $1 < j < N$. In this argument $\| \cdot \|$ can denote any norm. What is essential is that $P^{(n)} \to 0$ strongly. In particular one can use the norm of $H_{-1/2}$ provided that the system $\{\phi_j\}$ forms a basis of $H_{-1/2}$. Note that $H_{-1/2} \supset H$, so that if the system $\{\phi_j\}$ forms a basis of $H_{-1/2}$, then every element $f \in H$ can be represented in the form $f = \sum_{j=1}^\infty c_j \phi_j$, where the series converges in $H_{-1/2}$. It does not converge in H, generally speaking, but there exist bases such that if $f \in H$ then the above series converges in H. For example, such a basis is the basis consisting of the eigenfunctions of the operator A_0 (see also Lemma 1 below). If $\{\phi_j\}$ is a basis of H_q then $\{A_0^s \phi_j\}$ is a basis of H_{q+s}. This follows from the fact that A_0^s is a bijection of H_q onto H_{q+s} (A_0^s is an elliptic pseudodifferential operator of order $-s$). Since $|f|_q < |f|_s$ for $q < s$, it is clear that if the series $\sum_{j=1}^\infty c_j \phi_j$ converges to f in $H = H_0$ it converges to f in $H_{-1/2}$. It is convenient to have a system $\{\phi_j\}$ which forms a basis in any of H_q and if $f \in H_q$ the series $f = \sum_{j=1}^\infty c_j \phi_j$ converges in H_q. For example, if S^1 is the unit circle and $H_q = H_q(S^1)$ then the system $\{\exp(inx)/\sqrt{2\pi}\}$ forms a basis of H_q for any q. The same property has the system $\{\psi_j\}$ of the eigenfunctions of the Laplace–Beltrami operator on Γ, but practically this system is difficult to construct explicitly. Let us prove that for a starlike domain D the system $\{Y_j(\xi)\}$, where $\xi = (\theta, \phi)$ is a point on a unit sphere S^2 and Y_j are the normalized spherical harmonics, forms a basis in each of H_q. A domain is called starlike if there exists a point 0 inside the domain such that every point of the boundary of the domain can be seen from this point. This means that the equation of the boundary is of the form $r = R(\theta, \phi) = R(\xi)$, where the origin is at the point 0. It is well known that

$$Q_0 Y_n \equiv \int_{S^2} \frac{Y_j(\xi') d\xi'}{4\pi r_{\xi\xi'}} = \frac{Y_j(\xi)}{2n+1}, \quad j = 0,1,2,\ldots$$

where $r_{\xi\xi'} = |\xi - \xi'|$. The system $\{Y_j\}$, $j = 0,1,2,\ldots$, forms an orthonormal basis of $H = L^2(S^2)$ and in any $H_q(S^2)$ the scalar product in $H_q(S^2)$ can be defined as $(u, v)_q = (Q_0^{-q} u, Q_0^{-q} v)_0$, $H_0 = L^2(S^2)$, and $(Y_n, Y_m)_q = (2n+1)^q (2m+1)^q (Y_N, Y_m) = (2n+1)^q (2m+1)^q \delta_{nm}$, where δ_{nm} is the Kronecker delta.

Lemma 1: The system $\{Y_j(\xi)\}$ forms a Riesz basis of $H_q = H_q(\Gamma)$, provided that D is starlike, $\Gamma \subset C^\infty$, and the elements of H_q are considered as functions of $\xi \in S^2$.

Proof: Consider the eigenfunctions of the equation

$$\int_\Gamma \frac{\psi_n(s') ds'}{4\pi r_{ss'}} = \lambda_n \psi_n(s), \quad r_{ss'} = |s - s'|, \quad s \in \Gamma. \tag{12}$$

Since D is starlike one can rewrite this equation as

$$Q\Phi_n = \int_{S^2} \frac{\Phi_n(\xi') p_0(\xi') d\xi'}{4\pi |R(\xi) - R(\xi')|} = \lambda_n \Phi_n(\xi), \tag{13}$$

where $s = R(\xi)$ is the equation of the surface Γ in the spherical coordinates, $\xi = (\theta, \phi)$, $\Phi_n(\xi) = \psi_n(R(\xi))$, $ds = p_0(\xi) d\xi$, $p_0(\xi) > 0$, and $d\xi = \sin\theta \, d\theta \, d\phi$. The function $p_0(\xi) = |R'_\theta \times R'_\phi|$, where $\times$ denotes the vector product and $r = R(\theta, \phi)$ is the parametric equation of the surface Γ. The system $\{\Phi_j\}$ of the eigenfunctions of the operator Q defined in (13) forms an orthogonal basis of the weighted space $L^2(S^2, p_0(\xi))$. Since(**) $0 < c_1 < p_0(\xi) < c_2$ the normalized system $\{\Phi_j p_0^{-1/2}\}$ forms an orthonormal basis of $L^2(S^2)$. Therefore this system is an image of the system $\{Y_j\}$, $j = 0,1,2,\ldots$, under a unitary transformation of $L^2(S^2)$: $UY_j = p_0^{-1/2} \Phi_j$ or $\Phi_j = p_0^{1/2} UY_j$. The operator $p_0^{1/2} U$ is a bijection of $L^2(S^2)$ onto itself. Let us introduce the operator $J\Phi_j = \psi_j$, $j = 0,1,2,\ldots$, where ψ_j are the normalized eigenfunctions of Eq. (12). The operator J defined on the basis elements is isometric and can be extended to the isometric bijection $J: L^2(S^2) \to L^2(\Gamma) = H$. Therefore $\psi_j = J p_0^{1/2} UY_j$, $j = 0,1,2,\ldots$, and Lemma 1 is proved for $H = H_0$. For $q \neq 0$ the system Y_j forms an orthogonal basis of $H_q(S^2)$ and the space $H_q = H_q(\Gamma)$ is metrically equivalent to $H_q(S^2)$ because Γ is C^∞ diffeomorphic to S^2. Thus the system $\{Y_j\}$ forms a basis of $H_q(\Gamma)$ for any q. If instead of C^∞ diffeomorphism one assumes that Γ is C^l diffeomorphic to S^2, then $\{Y_j\}$ forms a

basis of $H_q^\cdot$, $q<l$. Lemma 1 is proved.

(3) Let us consider another projection method of solving Eq. (6) corresponding to the least squares method; namely

$$(Ah_n - v, A\phi_j) = 0, \quad 1<j<n, \tag{14}$$

or

$$\sum_{m=1}^{n} b_{jm} c_m = d_j, \quad 1<j<n, \tag{15}$$

where

$$b_{jm} = (A\phi_j, A\phi_m), \quad d_j = (v, A\phi_j). \tag{16}$$

Since (b_{jm}) is a positive define matrix (if $\ker A = \{0\}$ which we assume for simplicity), the system (15) is uniquely solvable for any n. This system can be obtained from the least squares method as a necessary condition of the minimum of the functional

$$\| Ah_n - v\|^2 = \min, \tag{17}$$

or

$$|(I+T)h_n - w|^2_{-l} = \min, \quad w = A_0^{-1} v. \tag{18}$$

Since $I+T$ is a bijection of H_{-l} onto itself and A_0^{-1} is a bijection of H_q onto H_{q-l} the solution of (18) tends to $(I+T)^{-1}w$ as $n \to \infty$ in H_{-l} if $v \in H_0$, and in H_{q-l} if $v \in H_q$.

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APPENDIX V

Convergence of the T-matrix approach in scattering theory II.

by

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Abstract

Convergence of the T-matrix scheme is proved under more general assumptions than in A.G. Ramm, J. Math. Phys. 23, 1123-1125 (1982), and for more general boundary conditions. Stability of the numerical scheme towards small perturbations of data and convergence of the expansion coefficients are established. Dependence of the rate of convergence on the choice of basis functions is discussed.

Dependence of the quality of expansions in various spherical waves on the shape of the obstacle is discussed.

PACS 03.80.+r, 11.20.-e, 02.30.Jr.

1. Introduction

1. Let D be a bounded obstacle with the boundary Γ. Consider the following problem:

$$(\nabla^2+k^2)u=0 \quad , \; k>0 \quad \text{in } \Omega, \tag{1}$$

$$u|_{\Gamma}=f, \tag{2}$$

$$r\left(\frac{\partial u}{\partial r}-iku\right)\to 0 \quad , \; r\to\infty, \tag{3}$$

where Ω is the exterior domain, and f is given. Later we discuss other boundary conditions than (2), but the basic arguments and conclusions will be similar to those for problem (1)-(3).

The corresponding scattering problem is as follows: find the solution to Eq. (1) satisfying boundary condition (2) with f=0 and of the form $u=u_0+v$, where v satisfies the radiation condition (3) and u_0 is the incident field. It is clear that this problem reduces to problem (1)-(3) for v with $f=-u_0$ on Γ. Therefore, we discuss in what follows problem (1)-(3). There is an extensive literature about this problem. The existence and uniqueness of the solution to this problem for Liapunov boundaries are established long ago and are available in textbooks now [1]. The case of nonsmooth boundaries was also treated [2]. Numerical methods for solving problem (1)-(3) are known (finite differences, see e.g. [3], numerical solution of the boundary integral equations of the second and first kind [4]).

Our concern is with the T-matrix scheme [5]. This numerical

scheme was widely used during the last decade in the problems of acoustic, electromagnetic, and elastic wave scattering by one and many bodies, for scattering from periodic structures etc. [5-10].

Nevertheless the basic questions concerning convergence of the scheme, stability of the numerical scheme towards small perturbations of the data remained open. In [11] these questions were discussed for the first time. Here the results from [11] are strengthened and extended.

2. Let us describe the T-matrix scheme in a general formulation. Let $\{\psi_n\}$ be a system of outgoing (not necessarily spherical) waves, i.e.

$$(\nabla^2+k^2)\psi_n = 0 \quad \text{in } \Omega, \tag{4}$$

$$r\left(\frac{\partial\psi_n}{\partial r} - ik\psi_n\right) \to 0\ , \quad r\to\infty. \tag{5}$$

From the Green's formula it follows that

$$\int_\Gamma \psi_n \frac{\partial u}{\partial N}\, dS = \int_\Gamma u \frac{\partial \psi_n}{\partial N}\, dS\ , \quad \forall n\ , \tag{6}$$

where u is the solution to (1)-(3) and N is the exterior unit normal on Γ (pointing out, into Ω). Using boundary condition (2) one writes (6) as

$$\int_\Gamma \psi_n h\, dS = f_n\ , \quad \forall n, \tag{7}$$

where

$$f_n \equiv \int_\Gamma f \frac{\partial\psi_n}{\partial N}\, dS\ , \tag{8}$$

$$h \equiv \frac{\partial u}{\partial N}\bigg|_{\Gamma}. \tag{9}$$

The T-matrix scheme consists of the following. Let $\{\phi_j\}$ be a linearly independent and complete system of functions in $H_0 = L^2(\Gamma)$. Let

$$h_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j \,, \tag{10}$$

where $c_j^{(m)}$, $1 \le j \le m$, are constant coefficients, which should be defined from the linear algebraic system

$$\sum_{j=1}^{m} a_{nj} c_j^{(m)} = f_n \,, \quad 1 \le n \le m \,, \tag{11}$$

$$a_{nj} = \int_{\Gamma} \psi_n \phi_j \, dS = (\phi_j, \overline{\psi_n}). \tag{12}$$

One obtains this system if (10) is substituted in (7) and only the first m equalities (7) are used.

3. Justification of the T-matrix scheme requires positive answers to the following questions:

Q1. Is (11) solvable for sufficiently large m?

Q2. Does $h_m \underset{H_0}{\to} h$, $m \to \infty$? Here h is defined as in (9).

Q3. Does $c_j^{(m)} \to c_j$, $m \to \infty$? Is the convergence uniform in j, $1 \le j < \infty$?

Q4. Does the equality $h = \sum_{j=1}^{\infty} c_j \phi_j$ hold, where c_j are defined in Q3, and it is assumed that the limits c_j exist?

Q5. How does the rate of convergence depend on the choice of the systems $\{\phi_j\}$, $\{\psi_n\}$?

Q6. Is the numerical scheme based on the equation (11) stable towards small perturbations of f_n and the matrix a_{nj}?

Remark 1. In the literature [12] the following questions were discussed: Is the set of equations (7) solvable? Is the solution to (7) unique? These questions are easy to answer. The set of equations (7) is solvable for any system $\{\psi_n\}$ satisfying (4) and (5): take the solution u to (1)-(3) (which does exist) and apply Green's formula to ψ_n and u to obtain (7). The solution to (7) is unique iff the system $\{\psi_n\}$ is closed in $H_0=L^2(\Gamma)$ so that

$$\int_\Gamma h\psi_n dS = 0, \ \forall n \Rightarrow h=0. \tag{13}$$

This is equivalent to saying that any $h\in H_0$ can be approximated with prescribed accuracy ε in the norm of H_0 by linear combinations of the elements ψ_n: $\| h-\sum_{j=1}^{m(\varepsilon)} c_j(\varepsilon)\psi_j\|<\varepsilon$, i.e. the system $\{\psi_j\}$ is complete. We assume below that the system $\{\psi_n\}$ is closed.

4. If one takes as ψ_1 in (7) $g(s,y)=g(s,y,k)=\frac{\exp(ik|s-y|)}{4\pi|s-y|}$ $y\in\mathcal{D}$, and does not use equations (7) for n>1, then one gets the integral equation

$$\int_\Gamma g(s,y)h(s)dS = \int_\Gamma f(s)\frac{\partial g(s,y)}{\partial N_s}dS \equiv F(y), \ y\in\mathcal{D}. \tag{14}$$

Actually, if (14) holds for $y\in B\subset\mathcal{D}$, where B is any ball lying strictly in $\mathcal{D}$, then (13) holds in $\mathcal{D}$ because both sides in (14) are solutions to the Helmholtz equation in $\mathcal{D}$ and therefore, if they are identical in B they are identical in $\mathcal{D}$. If one lets $y\to s'\in\Gamma$ one obtains from (14) the boundary integral equation of

the first kind

$$Ah = \int_\Gamma g(s,s')h(s)dS = b(s'), \tag{15}$$

where

$$b(s') = \lim F(y), \quad y \in D, y \to s'. \tag{16}$$

If one looks for a solution of (15) of the form (10) and uses a projection method for finding $c_j^{(m)}$, one obtains the system

$$\sum_{j=1}^{m} A_{nj} c_j^{(m)} = b_n \quad , 1 \le n \le m, \tag{17}$$

where

$$A_{nj} = (A\phi_j, \eta_n) \; ; \quad b_n = (b, \eta_n), \tag{18}$$

$$(f,h) \equiv \int_\Gamma f\bar{h}dS, \tag{19}$$

and the bar denotes complex conjugation. The same questions Q1-Q6 can be studied for system (17). In this case η_n plays the role of ψ_n, but now there is no need to assume anything about the properties of η_n in Ω. In fact, η_n are defined only on the surface Γ. Questions Q1, Q2, Q5, Q6 were answered in [11] for the system (17) under the assumption that $\eta_n=\phi_n$ and the system $\{\phi_n\}$ forms a basis of $H_{-\frac{1}{2}}$. The spaces $H_q=W_2^q(\Gamma)$, $-\infty<q<\infty$ are defined as the spaces of functions with q square integrable derivatives for $q\ge0$ integer, and as dual spaces (spaces with negative norm) for $q<0$. For arbitrary $q<0$ they can be defined as interpolating spaces, or directly [13]. In the present paper we note that the result and arguments in

[11] are valid under the weaker assumption that $\{\phi_j\}$ is a complete system of linearly independent functions (not necessarily a basis, see Sec. 2.3 below).

From the integral equation (14) one can go back to the system (7) by assuming that

$$g(s,y) = \sum_{j=1}^{\infty} \Phi_j(y)\psi_j(s), \quad |y|<|s|, \tag{20}$$

substituting (20) into (14) and equating coefficients in front of ϕ_j. From this point of view the integral equation (14) is equivalent to the system (7)-(8). In the literature expansion (20) is used with ψ_j being the outgoing spherical waves and Φ_j being the regular (i.e. finite at the origin) solutions to Helmholtz' equation. Matrix (18) will be identical to matrix (12) if ψ_n in (12) are chosen so that $A\bar{\eta}_n=\psi_n$. This corresponds to a specific choice of the outgoing waves, since for any linearly independent system of functions η_n in $L^2(\Gamma)$, the system $\psi_n \equiv A\bar{\eta}_n$ will be a system of outgoing waves whose boundary values on Γ form a linearly independent system in $L^2(\Gamma)$, provided that the operator $A: L^2(\Gamma) \to L^2(\Gamma)$ has no zeros (i.e. $A\eta=0 \Rightarrow \eta=0$). This will be the case iff k^2 is not an eigenvalue of the interior Dirichlet Laplacian in $\mathcal{D}$.

In [11] it was noted that in the case when k^2 is an eigenvalue of the Dirichlet Laplacian in $\mathcal{D}$, one can use, instead of $g(x,y,k)$, the Green's function $g_\varepsilon(x,y,k)$ and in this case the corresponding operator A will have no zeros. The function g_ε is the Green's function of the Dirichlet operator ∇^2+k^2 in the exterior of a small ball B_ε situated in $\mathcal{D}$, where B_ε is so chosen that k^2 is not an eigenvalue of the problem

$$(\nabla^2+k^2)u=0 \quad \text{in } \mathcal{D}\setminus B_\varepsilon\,, \quad u|_\Gamma=0, \quad u|_{\partial B_\varepsilon}=0$$

where ∂B_ε is the boundary of B_ε, and $\mathcal{D}\setminus B_\varepsilon$ is the complement in $\mathcal{D}$ to B_ε.

The above argument shows that the analysis in [11] is applicable to equation (11) under some special choice of ψ_n in (12).

2. Analysis of the T-matrix scheme

In this section we discuss questions Q1-Q6 formulated in Sec. 1.3.

1. The system (11) is solvable for a given m iff $\det(a_{nj})_{n,j=1}^{m} \neq 0$. For the following analysis we need some definitions and results from the theory of Hilbert spaces. These definitions and results are given in Appendix 1. In Appendix 3 some results about convergence and stability of projection methods are given.

We are interested in the properties of the coordinate systems $\{\phi_j\}$ and $\{\psi_j\}$ which imply positive answers to questions Q1, Q2, Q3, and Q6. Let us write equation (7) as an operator equation

$$ah = f, \quad a: H_0 \to \ell^2, \quad H_0 = L^2(\Gamma), \tag{7'}$$

where the operator a is bounded and defined on all of H_0 iff

$$\sum_{n=1}^{\infty} |(h, \overline{\psi}_n)|^2 \leq c_2^2 \int_\Gamma |h|^2 dS \quad , \forall h \in H_0, \; c_2 > 0.$$

If this inequality does not hold for all $h \in H_0$, but the system $\{\psi_n\}$ is a basis of H_0, then a is densely defined (i,e, its domain D(a) is dense in H_0). Indeed, in this case the biorthogonal system $\{\tilde{\psi}_n\}$ is also a basis of H_0 [17, p. 307], and any linear combination $\sum_{j=1}^{m} c_j \tilde{\psi}_j \in D(a)$.

The operator a transforms a function $h \in H_0$ into a sequence $(h, \overline{\psi}_n) = \int_\Gamma h \, \psi_n \, dS$, $1 \leq n < \infty$. The range of a is dense in ℓ^2 provided that for any sequence $\{d_n\} \in \ell^2$ the series $\sum_{n=1}^{\infty} d_n \psi_n(x)$ converges in H_0 and the system $\{\psi_n\}$ is ω-linearly independent, i.e. $\sum_{n=1}^{\infty} d_n \psi_n = 0 \Leftrightarrow d_n = 0, \forall n$. Indeed, suppose that $(ah, d) = 0, \forall h \in H_0$, where the parentheses denote the inner product in ℓ^2. Then $\int_\Gamma h \sum_{n=1}^{\infty} \bar{d}_n \psi_n dS = 0$. Since

$h \in H_0$ can be taken arbitrary it follows that $\sum_{n=1}^{\infty} \overline{d}_n \psi_n = 0$ and $d_n = 0, \forall n$. The operator a^{-1} will be bounded and defined on all of ℓ^2 iff $\|ah\| \geq c_1 \|h\|, \forall h \in H_0, c_1 > 0$. We use the same notation for the norms in H_0 and ℓ^2. This inequality can be written as

$$\sum_{n=1}^{\infty} |(h, \overline{\psi}_n)|^2 \geq c_1^2 \int_{\Gamma} |h|^2 dS \quad , \quad \forall h \in H_0, \ c_1 > 0.$$

Therefore a and a^{-1} are both bounded iff

$$c_1^2 \|h\|^2 \leq \sum_{n=1}^{\infty} |(h, \overline{\psi}_n)|^2 \leq c_2^2 \|h\|^2 \quad , \forall h \in H_0, \ c_1 > 0.$$

These inequalities hold iff the system $\{\psi_j\}$ forms a Riesz basis of H_0. Let us consider the truncated equations (7), i.e. the system (11) as a projection method for solving (7'). Namely, let Q_m be an orthoprojection in ℓ^2 defined by the formula $Q_m f = f^{(m)} = (f_1, \ldots, f_m, 0, 0, \ldots)$, and P_m be an orthoprojection in H_0 on the linear span of $(\phi_1, \ldots \phi_m)$. The system (11) can be written as

$$Q_m a P_m h_m = Q_m f \quad , \quad P_m h_m = h_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j \tag{11'}$$

This equation is of the type studied in Appendix 3 (see A3.2). Conditions (A3.4) and (A3.5) are necessary and sufficient for equation (11') (i.e. (11)) to be uniquely solvable for all sufficiently large $m > m_0$ and for the convergence

$$\|h_m - h\| \to 0 \quad , \ m \to \infty. \tag{21}$$

Conditions (A3.4) can be written in our case as

$$\sum_{n=1}^{m}\left|\sum_{j=1}^{m} a_{nj} c_j\right|^2 \geq c\int_{\Gamma}\left|\sum_{j=1}^{m} c_j\phi_j\right|^2 dS \quad , \forall m > m_0 \, , \; c > 0 \tag{22}$$

where $c_1 \ldots c_m$ are arbitrary constants, $a_{nj}=(\phi_j,\bar{\psi}_n)$. Condition (22) can be written as

$$c_j \bar{c}_{j'} a_{nj} \bar{a}_{nj'} \geq c\, c_j \bar{c}_{j'} (\phi_j, \phi_{j'})$$

where one should sum over repeated indices and the bar denotes complex conjugation. This last inequality means that the following matrix inequality holds

$$(aa^*)_m \geq c(\Phi)_m \quad , \quad \forall m > m_0 \, , \quad c > 0. \tag{22$_1$}$$

Here $(a)_m$ is the truncated matrix: $(a)_m=(a_{nj})_{n,j=1}^{m}$. $\Phi=(\phi_j,\phi_{j'})=$ $=\int_{\Gamma}\phi_j\phi_{j'}dS$ is the Gram matrix for the system $\{\phi_j\}$. If $\lambda(\Phi)\,(\Lambda(\Phi))$ denotes the minimal (maximal) eigenvalue of a self-adjoint matrix $\Phi \geq 0$, then (22_1) holds if for example

$$\inf_m \lambda((aa^*)_m) \geq \lambda > 0 \, , \quad \sup_m \Lambda((\Phi)_m) \leq \Lambda < \infty . \tag{22$_2$}$$

This condition is convenient from a practical point of view. The conclusion is as follows: if (22_2) holds, then the projection method (11) for solving equations (7) converges, i.e. equations (11) are uniquely solvable for sufficiently large $m>m_0$ and $\|h_m-h\| \to 0$ as $m\to\infty$, where h is the solution of (7). The second equality (22_2) holds for example if the system $\{\phi_j\}$ forms a Riesz basis of H_0. If we take $\phi_j=\bar{\psi}_j$ on Γ, then $a=\Phi$ and inequality (22_1) holds iff the first inequality (22_2) holds. Indeed, if

$a=\phi$, then (22_1) takes the form $(a^2)_m \geq c(a)_m$, $m>m_0$, $a=a^*$. This inequality holds iff the spectrum of $(a)_m$ is bounded away from zero by a positive constant. To see this, let us use the spectral theorem for the self-adjoint operator $a\geq 0$:

$$((a^2-ca)\phi,\phi)=\int_0^\infty (t^2-ct)\,d(E_t\phi,\phi),$$

where E_t is the resolution of the identity for a, $\min_{t\geq 0}(t^2-ct)=-\frac{c^2}{4}$, $t^2-ct\geq 0$ if $t\geq c$. Therefore the operator a^2-ca will be nonnegative for some $c>0$ iff $a\geq\alpha>0$ where α is a positive constant and in this case we can take $\alpha=c$ in the inequality (22_1). Note that $\alpha=\lambda^{\frac{1}{2}}$ where λ is the constant in (22_2). If the system $\{\phi_j\}$ is such that $\inf_m \lambda((\phi)_m)\geq\lambda>0$ (in particular, if it is a Riesz basis of H_0) then the conclusion is as above (after formula (22_2)).

Condition (A3.5) means that for $m>m_0$ the set of vectors $\{a_{nj}\}_{n=1}^m$, $1\leq j\leq m$ is linearly independent. That is

$$\det(a_{nj})_{j,n=1}^m \neq 0, \quad \forall m>m_0. \tag{23}$$

This condition follows from (22) (see Remark 1 in Appendix 3 and formula (22_2)).

If (22) holds then system (11) is uniquely solvable for all $m>m_0$, and the function (10), where $\{c_j^{(m)}\}$, $1\leq j\leq m$, is the solution of (11), converges in H_0 to the solution of (7). The rate of the convergence is given by (A3.8). This rate depends on the rate of convergence of $(I-P_m)h$ to zero, i.e. on the rate of approximation of the function $h=a^{-1}f$ by the linear combinations $\sum_{j=1}^m c_j\phi_j$. Stability of the solution towards small perturbations of the operator (i.e. of the matrix a_{nj}) and the right hand side f

(i.e. the sequence $\{f_n\}$) follow from the result 1) in Section 2 of Appendix 3. Indeed, consider the perturbed system

$$\sum_{j=1}^{m} (a_{nj}+b_{nj})\tilde{c}_j^{(m)} = f_n+\varepsilon_n\ , \quad 1\le n\le m\ , \tag{11"}$$

where b_{nj} and ε_n are the small perturbations of the operator a and the right hand side f, respectively. Let b_{nj} be sufficiently small in the sense that the operator b corresponding to the matrix b_{nj} is sufficiently small in the norm: $||b||<\delta$. Here $b\colon H_0\to\ell^2$ can be considered as an operator which is defined as follows. The system $\{\psi_n\}$ is perturbed: $\tilde{\psi}_n=\psi_n+\eta_n$. This perturbation generates the perturbation b of a by the formula $bh=(h,\bar{\eta}_n)$. The matrix b_{nj} is then defined as $(\phi_j,\bar{\eta}_n)$. If $||b||<\delta$ and $\delta<c$ where c is the constant in (22) (or A3.4), then according to the result 1) in Section 2 of Appendix 3 the perturbed system (11") will be uniquely solvable for all sufficiently large m and the corresponding $\tilde{h}_m=\sum_{j=1}^{m}\tilde{c}_j^{(m)}\phi_j$ will tend to $\tilde{h}=\tilde{a}^{-1}\tilde{f}=(a+b)^{-1}(f+f_\varepsilon)$, where $f_\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots)$. $||f_\varepsilon||<\varepsilon$. Thus

$$\|\tilde{h}-h\| = \|(a+b)^{-1}f - a^{-1}f + (a+b)^{-1}f_\varepsilon\| \le c'(\varepsilon+\delta) \tag{24}$$

The constant c' can be specified:

$$c' = \|(a+b)^{-1}\| + \|f\|\,\|a^{-1}\|\,\|(a+b)^{-1}\|. \tag{25}$$

Here we used the identity $(a+b)^{-1}-a^{-1}=-(a+b)^{-1}ba^{-1}$ and the estimate $||(a+b)^{-1}-a^{-1}||\le||(a+b)^{-1}||\cdot||a^{-1}||\cdot||b||$, which follows from the identity.

The estimates (21), (24), (25) and the above arguments give answers to Q1, Q2, Q5, and Q6. We now pass over to a discussion

of questions Q3, Q4.

Let us assume that

$$\{\phi_j\} \text{ is a complete minimal system in } H_0 \tag{26}$$

(For the definition of minimal systems and their properties used below, see Appendix 1.) Then there exists a unique biorthogonal system $\{\tilde{\phi}_j\}$, $(\phi_i, \tilde{\phi}_j) = \delta_{ij}$. Equation (11) can be written as $(a_m)c^{(m)} = f^{(m)}$. Its solution gives $h_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j$. Therefore

$$c_j^{(m)} = (h_m, \tilde{\phi}_j). \tag{27}$$

Since $||h_m - h|| \to 0$ we conclude that

$$c_j^m \to c_j \ , \quad m \to \infty \tag{28}$$

Thus (26) implies a positive answer to Q3. We have

$$|c_j^{(m)} - c_j| \leq \|h_m - h\| \|\tilde{\phi}_j\| \leq \|h_m - h\| \sup_j \|\tilde{\phi}_j\|. \tag{29}$$

Therefore the condition

$$\sup_j \|\tilde{\phi}_j\| \leq C < \infty \tag{30}$$

implies that convergence in (28) is uniform in j, $1 \leq j < \infty$. Condition (30) holds, e.g. if

$$\text{the system } \{\phi_j\} \text{ forms a Riesz basis of } H_0 \tag{31}$$

(see Appendix 1). Condition (31) implies also the positive answer to Q4. Indeed, if (31) holds then h can be written as

$$h = \sum_{j=1}^{\infty} \tilde{c}_j \phi_j , \tag{32}$$

and the coefficients $\tilde{c}_j$ are uniquely determined by the element h. On the other hand we know that

$$\|h_m - h\| \to 0 \quad , \quad m \to \infty \tag{33}$$

where

$$h_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j \quad , \quad c_j^{(m)} \to c_j . \tag{34}$$

First, we conclude that $\tilde{c}_j = c_j$ because

$$\tilde{c}_j = (h, \tilde{\phi}_j) = \lim (h_m, \tilde{\phi}_j) = \lim c_j^{(m)} = c_j . \tag{35}$$

Secondly, we see from (32) and (35) that the answer to Q4 is yes.

In the above analysis the basic assumptions were (22), (26), and (31) and we explained which of these imply positive answers to which of the basic questions Q1-Q6.

2. In this section we discuss the assumptions (26) and (31) and a particular case of the matrix a_{nj} for which the convergence analysis is straightforward. Note that (26) and (31) deal only with one of the systems. Assumption (22) deals with the "interaction" between the systems $\{\phi_j\}$ and $\{\psi_j\}$.

Assumption (26) holds if the smallest eigenvalue of the matrix $\phi_{ij} \equiv (\phi_i, \phi_j)$, $1 \le i,j \le m$ is bounded away from 0: $\lambda_m \ge \lambda > 0$ (see Appendix 1). Assumption (31) holds iff the matrix ϕ_{ij} defines a bounded and boundedly invertible operator on ℓ^2 (see Appendix 1). Since

this is a selfadjoint matrix, this will be the case iff

$$\Lambda_m \leq \Lambda < \infty \quad , \quad \lambda_m \geq \lambda > 0,$$

where $\Lambda_m(\lambda_m)$ is the maximal (minimal) eigenvalue of the matrix ϕ_{ij}, $1 \leq i,j \leq m$.

One can measure the "interaction" between $\{\phi_j\}$ and $\{\psi_j\}$ by the operator generated by the matrix $a_{nj}-\delta_{nj}=q_{nj}$. The assumptions

$$a_{nj}=\delta_{nj}+q_{nj} \quad , \quad \delta_{nj}=\begin{cases} 0 & n\neq j \\ 1 & n=j \end{cases} \quad , \quad q=(q_{nj}),\ 1\leq n,j<\infty \qquad (37_1)$$

is a compact operator on ℓ^2.

$$(I+q)x=0, \ x\in \ell^2 \Rightarrow x=0 \qquad (37_2)$$

are sufficient for the unique solvability of (11) for all sufficiently large $m>m_0$, and for the convergence in ℓ^2: $\|c^{(m)}-c\|\to 0$, $m\to\infty$, where $c^{(m)}=(c_1^{(m)}\ldots c_m^{(m)},0,0\ldots)$, $c=\tilde{a}^{-1}f$, $\tilde{a}$ is the operator on ℓ^2 with the matrix a_{nj}. Indeed, assumptions (37_1), (37_2) and Fredholm's alternative imply that $\tilde{a}^{-1}$ exists, is defined on all of ℓ^2 and is bounded. This fact and the special structure of $\tilde{a}$ imply the above statement about convergence. To see this, let us write (11) as $c^{(m)}+P_mqc^{(m)}=f^{(m)}$, where P_m is the orthoprojection in ℓ^2 onto the m-dimensional space of vectors with components $c_j=0$ for $j>m$. Since q is compact and $P_m\to I$ strongly in ℓ^2 one concludes that $\|q-P_mq\|\to 0$, $m\to\infty$. Therefore $\|(I+P_mq)^{-1}-(I+q)^{-1}\|\to 0$, $m\to\infty$. This proves the statement about convergence of $c^{(m)}$ to c.

If the system $\{\phi_j\}$ forms a basis of $H_0=L^2(\Gamma)$ and $f\in\ell^2$, then the solution of (7) is $h=\sum_{j=1}^{\infty} c_j\phi_j$ where c is the limit in ℓ^2 when $m\to\infty$ of the solutions $c^{(m)}$ to (11) and the solution to the equation

$\tilde{a}c=f$. In this case $(37_1) \Rightarrow (37_2)$ due to Fredholm's alternative and the uniqueness of the solution to the equation $\tilde{a}c=0$. Let us show that $\tilde{a}c=0 \Rightarrow c=0$. If the system $\{\phi_j\}$ forms a basis of H_0, then equations (7) and $\tilde{a}c=f$ are equivalent. But the homogeneous equations (7) have only the trivial solution if the system $\{\psi_n\}$ is closed in $H_0=L^2(\Gamma)$. (This assumption about $\{\psi_n\}$ is very natural and was made in the very beginning, see Remark 1 in the Introduction.) Therefore, $\tilde{a}c=0 \Rightarrow c=0$. As our analysis shows, the behaviour of the smallest and largest eigenvalue of the matrices (ϕ_i,ϕ_j), $(\phi_i,\tilde{\psi}_j)$ are of basic importance in an analysis of convergence and stability of the T-matrix scheme. For the operators of the form $a=I+q$, where q is compact, the projection scheme is discussed below in subsection 4. In this case the justification of the projection scheme can be easily obtained. In [11] the problem was reduced to a projection scheme for the equation of the above form $(I+T)h=f$, where T was compact.

3. Let us discuss briefly the results in [11] from our general point of view. The matrix analogous to A_{nj} in (18) in the paper [11] was of the form $A_{nj}=(A\phi_n,\phi_j)$, where the operator A was defined in (15). It was noted in [11] that $A=A_0(I+T(k))$ where $A_0>0$ in $H_0=L^2(\Gamma)$ and $T(k)$ is compact in H_q for any $-\infty<q<\infty$. Furthermore, $A_0: H_q \to H_{q+1}$ is a continuous linear bijection of H_q onto H_{q+1}. Therefore, $A_{nj}=[(I+T(k))\phi_n,\phi_j]$, where $[u,v] \equiv (A_0u,v)=(u,v)_{-\frac{1}{2}}$ and $(u,v)_q$ being the inner product in H_q. The operator $I+T(k)$ was assumed to be invertible. This can be done without loss of generality (see [11] and our argument in the end of section 1.4 above). If $\{\phi_j\}$ forms a complete set of linearly independent functions in $H_{-\frac{1}{2}}$ our general argument shows that the system (17) is uniquely solvable for sufficiently large m, and the answers to the

remaining questions Q2-Q6 are similar to the ones given above (see Sec. 2.4 below).

In particular, we have stability as $m\to\infty$ of the numerical scheme corresponding to system (17), with the matrix $(A\phi_j, \phi_n)$ and the operator A defined in (15), with respect to small perturbations of the matrix A_{nj} and $\{b_n\}$. In [11] it was assumed that the system $\{\phi_j\}$ forms a basis of $H_{-\frac{1}{2}}$. This assumption is weakened here: only completeness in $H_{-\frac{1}{2}}$ of the system $\{\phi_j\}$ is required. If a linearly independent system $\{\phi_j\}$ is complete in $H_0=L^2(\Gamma)$, it will be complete in $H_{-\frac{1}{2}}$. Indeed, H_0 is dense in $H_{-\frac{1}{2}}$. Therefore, for any $f\in H_{-\frac{1}{2}}$ one can find $f_\varepsilon \in H_0$ such that $\|f-f_\varepsilon\|_{-\frac{1}{2}}<\varepsilon$. If the system $\{\phi_j\}$ is complete in H_0, then one can approximate f_ε in H_0 by a linear combination: $\|f_\varepsilon - \sum_{j=1}^{m(\varepsilon)} c_j(\varepsilon)\phi_j\|_0<\varepsilon$. Since $\|u\|_0 \geq \|u\|_{-\frac{1}{2}}$ one concludes that $\|f-\sum_{j=1}^{m(\varepsilon)} c_j(\varepsilon)\phi_j\|_{-\frac{1}{2}}<\varepsilon$. This means that the system $\{\phi_j\}$ is dense in $H_{-\frac{1}{2}}$. (The same argument shows that this system will be dense in H_q for any $q\leq 0$.) This remark simplifies the argument in [11] in the case when we do not require that the system $\{\phi_j\}$ be a basis. In [11] a basis in $L^2(\Gamma)$ was constructed from the "distorted spherical harmonics" under the additional assumption that Γ is star-shaped (i.e. there exists a point in $\mathcal{D}$ from which every point of Γ can be seen).

4. Let us outline a proof of convergence of the projection method for solving the equation $u+Tu=f$ in a Hilbert space H under the assumptions that T is compact and $(I+T)^{-1}$ is bounded. The projection scheme is as follows: the approximate solution u_m is sought in the form $u_m = \sum_{j=1}^{m} c_j^{(m)}\phi_j$, where $\{\phi_j\}$ is a complete set of linearly independent elements in H. Let P_n denote the orthoprojection onto the linear span of $\{\phi_1,\ldots\phi_n\}$. The coefficients $c_j^{(m)}$ are to be found from the equations

$(u_m+Tu_m-f,\phi_j)=0$, $1\le j\le m$. These equations can be written as an operator equation $P_m u_m+P_m T u_m-P_m f=0$. But $P_m u_m=u_m$ and therefore $(I+P_m T)u_m=P_m f$, or $(I+T-r_m)u_m=P_m f$, where $r_m=(I-P_m)T$. Since the system $\{\phi_j\}$ is complete, $\|P_m f-f\|\to 0$ as $m\to\infty$ for any fixed $f\in H$. Therefore $I-P_m\to 0$ strongly. This and the compactness of T imply that $\|r_m\|\to 0$. Therefore the operator $I+T-r_m$ is boundedly invertible for sufficiently large m:

$$(I+T-r_m)^{-1}=\{(I+T)[I-(I+T)^{-1}r_m]\}^{-1}$$

$$=(1-Vr_m)^{-1}V=\sum_{j=0}^{\infty}(Vr_m)^j V \quad \text{if } \|Vr_m\|<1.$$

Here $V=(I+T)^{-1}$. One can estimate the rate of convergence of $u_m=(I+P_m T)^{-1}P_m f$ to $u=(I+T)^{-1}f$. Indeed

$$\|u_m-u\|\le\|(I+P_m T)^{-1}(P_m f-f)\|+\|[(I+P_m T)^{-1}-(I+T)^{-1}]f\|$$

$$\le\|(I+P_m T)^{-1}\|\,\|P_m f-f\|+\|(I-Vr_m)^{-1}V-V\|\,\|f\|$$

Let $\|V\|\le a$, $\|r_m\|\le\varepsilon_m$, and $a\varepsilon_m<1$, then

$$\|(I+P_m T)^{-1}\|\le\sum_{j=0}^{\infty}(a\varepsilon_m)^j a=\frac{a}{1-a\varepsilon_m},$$

$$\|(I-Vr_m)^{-1}V-V\|\le\frac{a^2\varepsilon_m}{1-a\varepsilon_m}.$$

These estimates show that the rate of convergence of u_m to u is determined by ε_m, a, and $\|P_m f-f\|$. The above argument is a particular case of a known general theory [14].

5. So far we have discussed the case of the Dirichlet boundary condition (2). If one has the Neumann boundary condition

$$\frac{\partial u}{\partial N}=f \quad \text{on } \Gamma, \tag{2'}$$

then our arguments are essentially the same: equations (6) lead to equations (7) with $h=u|_\Gamma$, $f_n=\int_\Gamma \psi_n f dS$, and now the role of ψ_n in equation (7) is played by the functions $\frac{\partial \psi_n}{\partial N}$.

However, the integral equation corresponding to this case will differ from (15). Indeed, in this case from (2') and the formula

$$\int_\Gamma g(s,y) \frac{\partial u}{\partial N} dS = \int_\Gamma u \frac{\partial g(s,y)}{\partial N} dS \quad , \quad y \in D ,$$

one obtains

$$\int_\Gamma \tilde{h} \frac{\partial g(s,y)}{\partial N} dS = \int_\Gamma g(s,y) f dS \equiv F(y) \quad , \quad y \in D , \tag{14'}$$

where $\tilde{h}=u|_\Gamma$.

Let $y \to s' \in \Gamma$ in the above equation. Then, using the known formula for the limit value on Γ of the potential of double layer one obtains

$$\tilde{h} = B\tilde{h} - 2\tilde{b} , \tag{15'}$$

where

$$B\tilde{h} \equiv 2 \int_\Gamma \tilde{h}(s) \frac{\partial g(s,s')}{\partial N_s} dS \quad , \quad \tilde{b}(s') = \lim_{y \to s'} F(y) . \tag{16'}$$

If one has the impedance boundary condition

$$-\frac{\partial u}{\partial N} + \gamma u = f \quad \text{on } \Gamma , \tag{2''}$$

then again from (6), one obtains (7) with $h=u|_\Gamma$, $f_n=\int_\Gamma f \psi_n dS$ and now the functions $-\frac{\partial \psi_n}{\partial N} + \gamma \psi_n$ play the role of ψ_n in equation (7). Completeness of all these systems in $H_0=L^2(\Gamma)$ for the case when

ψ_n are outgoing spherical waves, or any system for which the expansion

$$g(x,y) = \sum_{j=1}^{\infty} \phi_j(x)\psi_j(y) \quad , \quad |x|<|y| , \tag{38}$$

holds, is easily seen. For example, if

$$\int_\Gamma f\psi_n dS = 0 \quad , \quad \forall n, \tag{39}$$

then (38) and (39) imply:

$$\int_\Gamma g(x,s) f\, dS = 0 \quad , \quad x \in \mathcal{D}. \tag{40}$$

Therefore $u(x) \equiv \int_\Gamma g(x,s) f dS$ solves the equation $(\nabla^2+k^2)u=0$ in $\mathcal{D}$ and in Ω, and $u|_\Gamma=0$. This implies that $u=0$ in Ω. If $u=0$ in Ω and in $\mathcal{D}$ then $f=(\frac{\partial u}{\partial N})_+ - (\frac{\partial u}{\partial N})_-=0$. Here $+(-)$ denote the limit values on Γ from the interior (exterior).

3. Numerical experiments

The purpose of the numerical computations presented here is to test whether some commonly used complete set of functions, e.g., outgoing and regular spherical waves, also satisfies the assumptions made in the previous sections. Do they, for instance, form a Riesz basis? Are they good for expansion of functions on Γ?

1. Before answering these questions it might be illustrative to consider a simpler one-dimensional case, e.g.,

$$\phi_m(x) = \sqrt{\frac{2}{\pi}}\, e^{-qmx} \sin mx \quad , \quad m = 1,2,3,\ldots \quad , \quad x \in [0,\pi]. \tag{41}$$

This set of functions is a perturbation of the orthonormal basis $\sqrt{\frac{2}{\pi}}$ sin mx by factors e^{-qmx}. Here the constant q can be taken as a measure of the eccentricity of the object. The motivation for our choice of the model example will become clear from the considerations given after formulas (45). The Gram matrix of this model problem can be calculated analytically. We have

$$(\phi_m, \phi_n) = \pi^{-1} q (m+n) \left[1 - (-1)^{m+n} e^{-q\pi(n+m)} \right] \Big\{ \left[q^2 (m+n)^2 + (m-n)^2 \right]^{-1} - \left[q^2 (m+n)^2 + (m+n)^2 \right]^{-1} \Big\}. \tag{42}$$

We define the condition number for an operator A as $\kappa = \|A\| \cdot \|A^{-1}\|$.

Notice that the Gram matrix is always selfadjoint nonnegative and the finite Gram matrix is positive definite, provided

the functions $\{\phi_j\}$ are linearly independent. For a positive self-adjoint operator the norms $||A||$ and $||A^{-1}||$ can be calculated by the formulas $||A|| = \Lambda$, $||A^{-1}|| = \lambda^{-1}$, where $\lambda = \min_{t \in \sigma(A)} t$, $\Lambda = \max_{t \in \sigma(A)} t$, and $\sigma(A)$ is the spectrum of A. In this case

$$\kappa = \Lambda \lambda^{-1} \tag{43}$$

Numerical data seems to show that even after normalization, the perturbed system $\phi_m = \sqrt{\frac{2}{\pi}}\, e^{-qmx} \sin mx$, $m=1,2,3,\ldots$, does not form a Riesz basis of $L^2([0,\pi])$. The non-normalized system $\{\phi_j\}$ is not a Riesz basis because the necessary condition $0 < c \le \inf_{m \ge 1} ||\phi_m||$ for a system $\{\phi_m\}$ to form a Riesz basis is violated.

In Table 1 we give the condition number κ defined in equation (43) for both $\{\phi_m\}$ and the normalized functions $\{\phi_m / ||\phi_m||\}$. Three different tendencies are noticed:

1) An increase in condition number κ as the truncation size grows.
2) An increase in condition number κ as the eccentricity grows.
3) The normalized functions have smaller condition number as compared to non-normalized ones

2. We now consider the spherical waves, i.e.,

$$\left.\begin{aligned} \psi_n(X) &= h_\ell^{(1)}(kr)\, Y_n(\omega) \\ \mathrm{Re}\,\psi_n(X) &= j_\ell(kr)\, Y_n(\omega) \end{aligned}\right\} ,\quad r = |X| . \tag{44}$$

Here $h_\ell^{(1)}(kr)$ is a spherical Hankel function of the first kind and $j_\ell(kr)$ is a spherical Bessel function. The spherical harmonics $Y_n(\omega)$, where $\omega = (\theta, \phi)$ is a unit vector, and θ, ϕ are the angular spherical coordinates, are normalized in $L^2(S^2)$, where S^2 is the unit sphere (n is a multi-index $n = (\ell, m)$).

For large orders ($\ell \gg kr$) in ψ_n and $\mathrm{Re}\psi_n$ it is known [20] that

$$\psi_n(X) \sim -i(2\ell-1)!!(kr)^{-\ell-1} Y_n(\omega)$$

$$\mathrm{Re}\,\psi_n(X) \sim ((2\ell+1)!!)^{-1}(kr)^{\ell} Y_n(\omega). \tag{45}$$

For large orders ℓ the spherical waves are essentially a perturbation of the basis $Y_n(\omega)$ by a power of kr. This is the promised motivation for the choice of the model example in (41).

In the numerical data given below we have also included the spherical harmonics Y_n and the functions

$$\chi_n(X) = (kr)^{-\ell-1} Y_n(\omega), \tag{46}$$

which are solutions of the equation $\Delta u=0$ in Ω. The factor k in (46) was used in order for the factor kr to be dimensionless.

The Gram matrix for these four systems has matrix elements, which are double integrals over the unit sphere. We assume that the equation of the surface Γ can be written as $s=p(\omega)$, where $\omega \in S^2$, and p is a smooth invertible function, so that $\omega=p^{-1}(s)$. In our calculations the functions (44) and (46) were considered on Γ, i.e. as function of s, where $r=|s|$, $x=s$ on the surface Γ. If the bodies are axially symmetric then the matrix elements of the Gramians can be written as single integrals in θ, and for simplicity we choose the surface of a spheroid, i.e.

$$r(\theta) = (\sin^2\theta/a^2 + \cos^2\theta/b^2)^{-1/2}, \tag{47}$$

where a and b are the semi-axes of the spheroid (θ is the polar angle, so that b is the semi-axis along the axis of symmetry). The mirror symmetry $r(\theta)=r(\pi-\theta)$ implies that ℓ even and ℓ odd do not couple. Thus the elements of the Gram matrix are zeroes if $\ell+\ell'$ is odd. In this case we can change the enumeration of the columns and rows in the matrix so that it becomes a block diagonal matrix with two blocks. The size of the first block, which corresponds to the rows with even numbers, is $\frac{\ell_{max}+1}{2}$ if ℓ_{max} is odd, and $\frac{\ell_{max}}{2}+1$ if ℓ_{max} is even. The size of the second block, which corresponds to the rows with odd numbers, is $\frac{\ell_{max}+1}{2}$ if ℓ_{max} is odd, and $\frac{\ell_{max}}{2}$ if ℓ_{max} is even.

The numerical computations seem to indicate that neither of the systems $\{\psi_n\}$, $\{Re\psi_n\}$, or $\{\chi_n\}$ forms a Riesz basis of $H_0=L^2(\Gamma)$. However, $\{Y_n(p^{-1}(s))\}$ forms a Riesz basis of H_0 as has been proven earlier [11]. It is seen from Table 2 that the spherical waves $\{\psi_n\}$, $\{Re\psi_n\}$, and $\{\chi_n\}$ are not good for expansions in the sense that the condition number of the Gram matrix grows as the truncation size increases. Indeed, in this more realistic example the tendencies 1)-3) discussed above for the model problem seem to be valid, together with the additional observation

4) The systems $\{\psi_n\}$ and $\{\chi_n\}$ have smaller condition number than $\{Re\psi_n\}$.

The numerical data seems to indicate that the normalized functions should be used for expansions of the surface field since the corresponding Gram matrix has a smaller condition number. Furthermore, there is an indication that for high truncations the normalized systems $\{\psi_n\}$ and $\{\chi_n\}$ are better than $\{Re\psi_n\}$. However, the difference is not very considerable. A

large condition number means that the Gram matrix is difficult to invert numerically. It also means numerical instability, i.e. strong dependence of the numerical results on the roundoff errors and errors in the data.

It should be noted that the Gram matrix is identical to a Q-matrix [6] in the T-matrix scheme for a special choice of expansion functions. Thus, some of the properties mentioned above might appear within this scheme for this special choice of the expansion functions. It should also be noted that when the T-matrix scheme is used to compute the scattered field, additional operators enter, which tend to improve the situation. However, a discussion of these aspects requires further investigation.

4. Concluding remarks

There are several questions, raised by the T-matrix approach to scattering, which require further study. We conclude by giving a short list of such questions.

1) Try the system $v_n(x) = \int_\Gamma \frac{\exp(ik|x-s|)}{4\pi|x-s|} \phi_n(s)\,dS$, $x \in \Omega$

of outgoing waves for calculations. Here the system $\{\phi_n\}$ is a complete system in $H_0 = L^2(\Gamma)$. If the system $\{\phi_n\}$ forms a Riesz basis of $H_{-\frac{1}{2}}$, then the system $\{v_n\}$ on Γ forms a Riesz basis of H_0. This was established in [11]. From the results in [11] it follows that any solution of the problem (1)-(3) can be represented by the series $u = \sum_{n=1}^{\infty} c_n v_n(x)$, which converges in Ω up to the boundary. Indeed, assume (without loss of generality) that k^2 is not an eigenvalue of the Dirichlet Laplacian in D. Then the solution to the problem (1)-(3) can be written as $u(x) = \int_\Gamma \frac{\exp(ik|x-s|)}{4\pi|x-s|}\sigma(s)dS$, where σ is the unique solution to the equation $\int_\Gamma \frac{\exp(ik|s'-s|)}{4\pi|s'-s|}\sigma(s')dS = f(s)$. If $\{\phi_n\}$ forms a basis of $L^2(\Gamma)$, then $\sigma = \sum_{n=1}^{\infty} c_n \phi_n$, where the series converges in $L^2(\Gamma)$. Therefore $u(x) = \sum_{n=1}^{\infty} c_n v_n(x)$, where the series converges uniformly in the closure of Ω. This was the reason for suggesting the system $\{v_n\}$ instead of the usual outgoing waves (44), which do not seem to form a Riesz basis on non-spherical surfaces. The other reason for choosing $\{v_n\}$ was that since the expansion of the solutions to Helmholtz' equation in the exterior domain in the functions v_n converges up to the boundary, no difficulties with the Rayleigh hypothesis arise.

2) It was noted in Section 3.2 that the Gram matrices correspond (for a particular choice of expansion systems) to a Q-matrix. More

extensive numerical experiments concerning the condition number for various matrices $a_{ij} \equiv (\phi_i, \bar{\psi}_j)$ (i.e. Q-matrices) are called for. This would provide a better basis for judging the performance of specific choices ϕ_i, ψ_j and thus provide a more detailed answer to Q5.

3) In the present article we have concentrated on "the null field equations" [5] (i.e. (7), (11) etc.) and the question of obtaining a solution on the surface Γ. In the T-matrix scheme one furthermore computes the (truncated) transition matrix $(T)_m$ of the form

$$(T)_m = (Q')_m \cdot [(Q'')_m]^{-1} \tag{48}$$

where the $m \times m$ matrices $(Q')_m$, $(Q'')_m$ are similar to the Q^1, Q^2 matrices in Appendix 2. The exact (infinite) T-matrix is independent of the expansion systems used on Γ. However, the approximate truncated matrix, computed according to (48), does contain such a dependence. It is of interest to investigate the rate of convergence of truncated forms like (48) to the true, infinite transition matrix for the scatterer in question. It is then of interest to exploit general constraints on the scattering matrix such as unitarity and symmetry (see e.g. [7] and the contribution by P.C. Waterman in [5]).

4) Extend the discussion of [11] and the present work to the case of a penetrable scatterer. Of particular relevance here is the relation between the convergence rates in the expansion used for the surface fields and their normal derivatives (one aspect of this relation is treated in Appendix 2).

5) Study the convergence questions for the T-matrix scheme for scattering from obstacles with noncompact (infinite) boundaries. In this context the work in [21] will be useful. In [21] the scattering problem was formulated and solved for domains with non-compact boundaries. For the boundaries, which are locally Liapunov and such that outside of a sphere of arbitrarily large but fixed radius the points of the boundary can be seen from a point located outside the domain $\mathcal{D}$ (in this case infinite), it was proved in [21] that the Schrödinger operator with decreasing real-valued potential has no positive discrete spectrum and the radiation condition selects a unique solution to the Dirichlet boundary value problem. The case of the third boundary condition was also treated. Furthermore, the existence, uniqueness, and properties of the resolvent kernel G of the Schrödinger operator were studied in detail. In particular, global estimates of $G(x,y,k)$ and its derivatives, uniform in $a \leq k \leq C$, $0<a<C<\infty$, were obtained for $|x-y| \to 0$ and $|x-y| \to \infty$. It was proved that the limit $G(x,y,k+i\varepsilon)$ as $\varepsilon \to 0$, $\varepsilon>0$, does exist and is attained uniformly in $a \leq k \leq C$. This was done for the boundaries Γ for which $\rho(s,\Gamma_0)\ (1+|s|^{\alpha}) \to 0$ as $|s| \to \infty$, $s \in \Gamma$. Here $\alpha>n$, where n is the dimension of the space, and Γ_0 is the boundary of the "canonical" domain (the boundary of a cone if $n>2$, and of a wedge in the two-dimensional case). It was proved [21] that $G(x,y,k) = \frac{e^{ikr}}{4\pi r} u(\nu,y,k) \cdot$ $\cdot (1+o(1))$ as $|x|=r \to \infty$, $xr^{-1}=\nu$, where the functions $u(\nu,y,k)$ are the solutions of the scattering problem in the sense that they solve the Schrödinger equation and vanish on Γ. Furthermore, it was shown that an arbitrary function $f \in L^2(\Omega)$ can be expanded in a Fourier integral in functions u and a Fourier series corresponding to the negative discrete spectrum of the Schrödinger operator. If the potential is

equal to zero then the Fourier series part of the expansion is absent. The wave operators were constructed in [21] with the help of the eigenfunction expansions.

Acknowledgement

This paper was written while the second author was visiting the Institute of Theoretical Physics, Chalmers University of Technology in Göteborg during August 1982. He is grateful to the Institute for its hospitality and to Dr. S. Ström and his group for a stimulating atmosphere which they created during his visit.

Appendix 1: Some results from linear functional analysis.

1. The gap of subspaces of a Hilbert space and a condition for invertibility of the mixed Gram matrices (ϕ_i, ψ_j).

Let H_1 and H_2 be (closed) subspaces of a Hilbert space H. Then the gap of H_1 and H_2 is defined as

$$\theta(H_1, H_2) = \|P_1 - P_2\|,$$

where P_1 and P_2 are the orthoprojections onto H_1 and H_2, respectively. Clearly $0 \le \theta \le 1$, $\theta(H_1, H_2) = \theta(H_2, H_1)$. It can be proved [14] that

$$\theta(H_1, H_2) = \max\Big\{ \sup_{\substack{\|x\|=1 \\ x\in H_1}} \|(I-P_2)x\|, \sup_{\substack{\|x\|=1 \\ x\in H_2}} \|(I-P_1)x\| \Big\}.$$

The following facts hold ([14], p. 252-260).

<u>Lemma 1</u>. Let G_n and H_n be n-dimensional subspaces of a Hilbert space H and $\theta(G_n, H_n) < 1$. Then G_n and H_n have orthogonal bases $u_1, \ldots u_n$ and $v_1 \ldots v_n$, respectively, and $(u_i, v_j) = \beta_i \delta_{ij}$, $1 \le i, j \le n$, where

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \ne j \end{cases}, \quad \{1 - \theta^2(G_n, H_n)\}^{1/2} \le \beta_i \le 1 .$$

<u>Lemma 2.</u> Let $\{\phi_j\}$ and $\{\psi_j\}$, $1 \le j \le m$, be two sets of linearly independent elements of H. Let $0 < \lambda_m$ and $0 < \mu_m$ denote the smallest eigenvalues of the matrices (ϕ_i, ϕ_j) and (ψ_i, ψ_j), $1 \le i, j \le m$, respectively. Let G_m and H_n be linear spans of $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_m$, respectively, and $A_m = (\phi_i, \psi_j)$, $1 \le i, j \le m$. Let $\theta(G_m, H_m) < 1$. Then A_m is invertible and

$$\|A_m^{-1}\| \le \frac{\alpha_m}{(\lambda_m \mu_m)^{1/2}},$$

where

$$\alpha_m = \{1-\theta^2(G_m, H_m)\}^{1/2}$$

2. Minimal systems.

A system of elements is called minimal if none of the elements belongs to the closure of the linear span of the others. A minimal system $\{\phi_j\}$ is called strongly minimal if $\lim_{m\to\infty}\lambda_m=\lambda>0$, where λ_m is the minimal eigenvalue of the Gram matrix $\phi_{ij}=(\phi_i,\phi_j)$, $1\le i,j\le m$. A system $\{\tilde{\phi}_j\}$ is called biorthogonal to the system $\{\phi_j\}$ if $(\tilde{\phi}_j,\phi_i)=\delta_{ij}$. The biorthogonal system $\{\tilde{\phi}_j\}$ is uniquely defined iff the system $\{\phi_j\}$ is minimal.

A system $\{\phi_j\}$ of linearly independent elements of a Hilbert space H is called closed in this space iff for $f\in H$ the conditions $(f,\phi_j)=0$, $\forall j$, imply that $f=0$.

A system $\{\phi_j\}$ of linearly independent elements of H is called complete in H iff for any $f\in H$ and any given positive number $\varepsilon>0$ one can find an element $\sum_{j=1}^{m(\varepsilon)} c_j\phi_j$ such that $\|f-\sum_{j=1}^{m(\varepsilon)} c_j\phi_j\|<\varepsilon$. Here the constants c_j and the number $m(\varepsilon)$ depend on ε and f.

A system $\{\phi_j\}$ can be complete but not forming a basis of H (see n.3 below for definitions of bases). Example: $H=L^2([0,1])$, $\phi_j=x^j$, $0\le j<\infty$. This system is complete in H, but is not a basis of H. Completeness follows from the Weierstrass approximation theorem. The fact that the system $\{x^j\}$ is not a basis of H can also be easily explained. Suppose that $\{x^j\}$ is a basis of H. Then (see n.3 below) for any $f\in H$ the series $f=\sum_{j=0}^{\infty} c_j x^j$ converges in $L^2([0,1])$. Therefore f is analytic in $|x|<1$ and cannot be an arbitrary element of H. In fact, from Müntz's theorem [16] it follows

that the system $\{\phi_j\}$ is not even minimal (every Schauder's basis is a minimal system). The Müntz's theorem says that the system $\{x^{p_j}\}$, $p_0=0$, $0<p_1<p_2<\ldots$ is complete in $L^2([0,1])$ iff there exists an infinite subsequence p'_j such that $\sum_{j=1}^{\infty} \frac{1}{p'_j}=\infty$.

3. Bases

A system $\{\phi_j\}$ is called a Schauder basis of a Banach space X if any element $x \in X$ can be uniquely represented as $x=\sum_{j=1}^{\infty} c_j \phi_j$, where the series converges in the norm of X. A system $\{\phi_j\}$ is called a Riesz basis of a Hilbert space H iff $\phi_j=Th_j$, where $\{h_j\}$ is an orthonormal basis of H and T is a linear bounded and boundedly invertible operator (i.e. T^{-1} is bounded and defined on all of H). A system $\{\tilde{\phi}_j\}$ biorthogonal to a basis $\{\phi_j\}$ of H is also a basis of H.

A complete system $\{\phi_j\}$ in H is a Riesz basis of H iff the matrix $\phi_{ij}=(\phi_i,\phi_j)$, $1\le i,j<\infty$, generates a bounded and boundedly invertible operator on ℓ^2.

A complete system $\{\phi_j\}$ in H is a Riesz basis of H iff there exist positive constants a_1, a_2 such that

$$a_1^2 \sum_{j=1}^{m} |c_j|^2 \le \|\sum_{j=1}^{m} c_j\phi_j\|^2 \le a_2^2 \sum_{j=1}^{m} |c_j|^2 ,$$

for any m and any constants c_j, $1\le j\le m$. If $\{\phi_j\}$ is a Riesz basis then $\sup_{1\le j<\infty} \|\phi_j\| \le a_2$, $\inf_{1\le j<\infty} \|\phi_j\| \ge a_1$ and similar inequalities hold for $\|\tilde{\phi}_j\|$. In particular $\sup_{1\le j<\infty} \|\tilde{\phi}_j\| <\infty$. These results can be found e.g. in [16], [17], [19]. In [15] the notion of the Riesz basis with brackets is applied to some nonselfadjoint integral equations arising in scattering theory.

4. Tests for boundedness and compactness of a linear operator on ℓ^2.

Let (a_{ij}), $1 \le i,j < \infty$, be a matrix which is considered as a linear operator A on ℓ^2. When is A bounded and when is A compact?

$(Ax)_i = a_{ij}x_j$

Here and below we sum over the repeated indices.

$$\|Ax\|^2 = a_{ij}x_j\overline{a_{ik}}\overline{x}_k \le |a_{ij}\overline{a}_{ik}| \frac{|x_k|^2+|x_j|^2}{2} \le \sup_{1\le k<\infty} \sum_{j=1}^{\infty} |a_{ij}a_{ik}| \, \|x\|^2.$$

Therefore

$$\|A\| \le \sup_{1\le k<\infty} \Big(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij}a_{ik}| \Big)^{1/2} \tag{A1.1}$$

The operator A is compact if $\sup_{1\le k<\infty} \big(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |a_{ij}a_{ik}| \big)^{\frac{1}{2}} < \infty$ and

$$\sup_{1\le k<\infty} \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} |a_{ij}a_{ik}| \longrightarrow 0 \ , \quad N \longrightarrow \infty . \tag{A1.2}$$

<u>Lemma</u> (Schur): Let $a_{ij} = \overline{a}_{ji}$ and $\sup_i \sum_j |a_{ij}| \le M$. Then $\|A\| \le M$, $A: \ell^2 \to \ell^2$.

<u>Proof</u>: It suffices to prove that $|(Ax,x)| \le M\|x\|^2$. One has

$$|(Ax,x)| \le |a_{ij}x_j\overline{x}_j| \le |a_{ij}| \frac{|x_i|^2+|x_j|^2}{2} \le \sum_j |a_{ij}|\,|x_i|^2 \le M\|x\|^2$$

5. Spaces with negative norms [13].

Let H_+ and H be Hilbert space $H_+ \subset H$ and H_+ is dense in H. Let $u \in H_+$, $f \in H$. Consider the completion H_- of H in the norm

$$\|f\|_- = \sup_{\substack{u \ne 0 \\ u \in H_+}} \frac{|(f,u)|}{\|u\|_+},$$

where $\|\cdot\|_+$ denotes the norm in H_+ and (f,u) denotes the inner product in H. The space H_- is a Hilbert space, $H_+ \subset H \subset H_-$, and H is dense in H_-.

6. Bessel and Riesz-Fischer systems. Interpolation in Hilbert space.

The basic equations (7) can be considered as an interpolation problem in the Hilbert space H ($H=H_0=L^2(\Gamma)$ in our case), i.e. the problem $(h,\psi_n)=f_n$, $n=1,2,\ldots$.

Definition 1: A system $\{\psi_n\}$ is called a Bessel system if $\sum_{n=1}^{\infty}|(h,\psi_n)|^2<\infty$, whenever $h\in H$.

Definition 2: A system $\{\psi_n\}$ is called a Riesz-Fischer system if the problem

$$(h,\psi_n)=f_n \quad , \ n=1,2,3,\ldots , \tag{A1.3}$$

is solvable whenever $\{f_n\}\in\ell^2$.

The set of sequences $\{(h,\psi_n)\}$, $n\geq 1$, $h\in H$ is called the moment space M of $\{\psi_n\}$. The questions of interest are:

1) When does a sequence $\{f_n\}\in M$? That is, when is the problem $(h,\psi_n)=f_n$, $n\geq 1$, solvable?
2) Is the solution unique?
3) How to construct the solution?

The answer to question 1 is given by

Proposition 1. In order that (A1.3) be solvable and $||h||\leq C$ it is necessary and sufficient that

$$\left|\sum_{n=1}^{m} a_n \bar{f}_n\right| \leq C \left\|\sum_{n=1}^{m} a_n\psi_n\right\|$$

for any m and any scalars a_n.

The answer to question 2 was already given: the solution of the problem (A1.3) is unique iff the system $\{\psi_n\}$ is closed in H. The following facts are useful [19].

Proposition 2. If (A1.3) is solvable then it has a unique solution of minimal norm.

Proposition 3. The solution of

$$(h, \psi_n) = f_n \quad , \quad n \leq m \tag{A1.4}$$

of minimal norm always exists, is unique, and can be found by the formula

$$h_m = -\{\det(\psi_i, \psi_j)\}^{-1} \det \begin{pmatrix} 0 & \psi_1 & \cdots & \psi_m \\ f_1 & (\psi_1, \psi_1) & \cdots & (\psi_1, \psi_m) \\ \vdots & & & \\ f_m & (\psi_m, \psi_1) & \cdots & (\psi_m, \psi_m) \end{pmatrix} \tag{A1.5}$$

Moreover, if (A1.3) is solvable and h is its unique solution of minimal norm then

$$\|h_m - h\| \to 0 \quad , \quad m \to \infty . \tag{A1.6}$$

Appendix 2. Discussion of some expansions occuring in the T-matrix scheme for a penetrable scatterer

Using the Green's theorem one obtains the identity

$$0 = \int_{\Gamma} \left(\mathrm{Re}\psi_n \frac{\partial u_+}{\partial N} - u_+ \frac{\partial \mathrm{Re}\psi_n}{\partial N} \right) dS \,, \quad \forall n \tag{A2.1}$$

where + denotes the limit value from the interior (see (44) for the definition of $\mathrm{Re}\psi_n$). Assuming that the following series converge in $L^2(\Gamma)$

$$u_+ = \sum_n c_n^{(1)} \psi_n \,, \tag{A2.2}$$

$$\frac{\partial u_+}{\partial N} = \sum_n d_n^{(1)} \frac{\partial \psi_n}{\partial N} \,, \tag{A2.3}$$

one substitutes the series in (A2.1) to obtain

$$0 = \sum_{n'} \left\{ d_{n'}^{(1)} \int_{\Gamma} \mathrm{Re}\psi_n \frac{\partial \psi_{n'}}{\partial N} dS - c_{n'}^{(1)} \int_{\Gamma} \psi_{n'} \frac{\partial \mathrm{Re}\psi_n}{\partial N} dS \right\} , \quad \forall n. \tag{A2.4}$$

Truncating the series and using the first m equations in (A2.4) one obtains a linear system, which can be solved for $c_n^{(1)}$ or $d_n^{(1)}$.

In the case when ψ_n and $\mathrm{Re}\psi_n$ are defined as in (44), it follows from the formula

$$\int_{\Gamma} \left[\mathrm{Re}\psi_n \frac{\partial \psi_{n'}}{\partial N} - \psi_{n'} \frac{\partial \mathrm{Re}\psi_n}{\partial N} \right] dS = \lambda \delta_{nn'} \tag{A2.5}$$

that the truncated linear system can be written as

$$Q^{(1)} \vec{d}^{(1)} - \left(Q^{(1)} - \lambda I \right) \vec{c}^{(1)} = 0 \tag{A2.6}$$

where $\vec{d}^{(1)} = (d_1^1 \ldots, d_m^1)$, $\vec{c}^{(1)} = (c^1 \ldots c_m^1)$ and

$$Q^{(1)}_{nn'} \equiv \int_{\Gamma} \mathrm{Re}\,\psi_n \frac{\partial \psi_{n'}}{\partial N}\, dS \quad , \; n,n' \leq m \qquad \text{(A2.7)}$$

(λ is a known scalar, depending on the normalization of ψ_n and $\mathrm{Re}\psi_n$). Term by term differentiation of (A2.2) would imply $\vec{c}^{\,(1)}=\vec{d}^{\,(1)}$, which is not consistent with (A2.6). That termwise differentiation can <u>not</u> be used to obtain $\vec{d}$ (by taking $d_n^{(1)}=c_n^{(1)}$ in (A2.2)-(A2.3)) is most easily seen from the following example in which Γ is a sphere of radius a and in which one keeps only one term in the series (A2.2), so that the questions about convergence of the series are irrelevant. Take $u_+=h_n^{(1)}(ka)Y_n(\omega)$. Since $(\nabla^2+u^2)u_+=0$ for $|x|\leq a$ and $u|_{r=a}=h_n^{(1)}(ka)Y_n(\omega)$, one finds that $u_+(x)=\frac{h_n^{(1)}(ka)}{j_n(ka)}j_n(kr)Y_n(\omega)$ (we assume that $j_n(ka)\neq 0$). Therefore

$$\frac{\partial u_+}{\partial N}\Big|_{r=a} = \frac{k h_n^{(1)}(ka)}{j_n(ka)} j_n'(ka)Y_n(\omega) \neq \frac{\partial}{\partial N} h_n^{(1)}(kr)Y_n(\omega)\Big|_{r=a} = k h_n^{(1)'}(ka)Y_n(\omega).$$

Indeed

$$\frac{h_n^{(1)}(ka)\,j_n'(ka)}{j_n(ka)} - h_n^{(1)'}(ka) = [j_n(ka)]^{-1}\left[h_n^{(1)}(ka)j_n'(ka) - h_n^{(1)'}(ka)j_n(ka)\right] \neq 0.$$

The numerator is the Wronskian of $h_n^{(1)}$ and j_n and is not zero. The general explanation is that even if one can continue the series (A2.2) analytically inside $\mathcal{D}$ in a neighbourhood of Γ, this continuation <u>will be different from</u> the solution u of the problem

$$(\nabla^2+k^2)u=0 \quad \text{in } \mathcal{D}, \quad u|_{\Gamma}=u_+, \qquad \text{(A2.8)}$$

so that $\frac{\partial u_-}{\partial N} \neq \frac{\partial w}{\partial N}$ on Γ, where w denotes the analytic continuation of the series (A2.2) in $\mathcal{D}$. Indeed, $u_+=w$ on Γ and if $\frac{\partial u_+}{\partial N}=\frac{\partial w}{\partial N}$ on Γ then $u\equiv w$ in $\mathcal{D}$ by the uniqueness of the solution to the Cauchy

problem for Helmholtz' equation. But this leads to a contradiction since w is singular at the origin ($0 \in \mathcal{D}$).

If, instead of (A2.2) and (A2.3), we consider the expansions

$$u_+ = \sum_n c_n^{(2)} \operatorname{Re}\psi_n \tag{A2.9}$$

$$\frac{\partial u_+}{\partial N} = \sum_n d_n^{(2)} \frac{\partial \operatorname{Re}\psi_n}{\partial N}, \tag{A2.10}$$

then the truncated matrix equation analogous to (A2.6) reads

$$Q^{(2)} \vec{d}_n^{(2)} = Q^{(2)} \vec{c}^{(2)}, \tag{A2.11}$$

where

$$Q_{nn'}^{(2)} \equiv \int_\Gamma \operatorname{Re}\psi_n \frac{\partial \operatorname{Re}\psi_{n'}}{\partial N}\, dS. \tag{A2.12}$$

Thus, assuming that $\{Q_{nn'}^{(2)}\}$, $n,n' \leq m$, is invertible, we obtain

$$\vec{c}^{(2)} = \vec{d}^{(2)} \tag{A2.13}$$

(this result is used in the T-matrix apparoach to scattering from a permeable body [6], [7]).

A relevant fact of more general nature is the following lemma.

Lemma Let

$$(\nabla^2 + k^2) u = 0 \quad \text{in } \mathcal{D}, \quad k > 0$$

$$(\nabla^2 + k^2) \phi_j = 0 \quad \text{in } \mathcal{D}, \quad 1 \leq j \leq m.$$

$$u_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j, \quad \frac{\partial u_m}{\partial N} = \sum_{j=1}^{m} c_j^{(m)} \frac{\partial \phi_j}{\partial N}.$$

Assume that

$$\left\{ (\nabla^2 + k^2) u = 0, \left. \frac{\partial u}{\partial N} \right|_\Gamma = 0 \right\} \Rightarrow u \equiv 0 \text{ in } \mathcal{D},$$

(so that the Green's function G_N for the interior Neumann problem

exists and is unique) and that the $\{c_j^{(m)}\}$, $j\leq m$, have been determined so that (where $\|\cdot\|$ is the norm in $H_0=L^2(\Gamma)$)

$$\left\|\frac{\partial u}{\partial N}-\frac{\partial u_m}{\partial N}\right\|<\varepsilon \tag{A2.14}$$

Then

$$\|u-u_m\|<c\varepsilon \tag{A2.15}$$

Here and below c denotes various constants depending on Γ.

Proof: Let $u-u_m\equiv v$. Then $(\nabla^2+k^2)v=0$ in $\mathcal{D}$ and $\left\|\frac{\partial v}{\partial N}\right\|<\varepsilon$. We have

$$v(x)=\int_\Gamma\left[G_N(x,s')\frac{\partial v}{\partial N'}-v\frac{\partial G_N(x,s')}{\partial N'}\right]dS'$$

$$=\int_\Gamma G_N(x,s')\frac{\partial v}{\partial N'}dS', \quad x\in\mathcal{D},$$

and also

$$v(s)=\int_\Gamma G_N(s,s')\frac{\partial v}{\partial N'}\,dS', \quad s\in\Gamma.$$

The following estimate is known [1]

$$|G_N(s,s')|\leq\frac{c}{|s-s'|},$$

With $h\equiv\frac{\partial v}{\partial N}$ we then have

$$|v(s)|\leq c\int_\Gamma\frac{|h|\,dS'}{|s-s'|},$$

which implies

$$\|v\|\leq c\varepsilon$$

since the operator S: $H_0\to H_0$, $(H_0=L^2(\Gamma))$

$$(Sh)(s)\equiv\int_\Gamma\frac{h(s')dS'}{|s-s'|}$$

is bounded. In fact, also S: $H_0\to H_1$ is bounded. Therefore

$$\left\|\frac{\partial v}{\partial s}\right\| \leq C\varepsilon$$

where $\frac{\partial v}{\partial s}$ is any tangential derivative of v. In the above it was essential that the $c_j^{(m)}$, $1 \leq j \leq m$, were determined so that (A2.14) was valid. If, instead, the $c_j^{(m)}$, $j \leq m$ are chosen so that

$$\|u - u_m\| < \varepsilon \tag{A2.16}$$

and if we assume in this case that

$$\{(\nabla^2+k^2)u = 0,\ u|_\Gamma = 0\} \Rightarrow u \equiv 0 \ \text{ in } \mathcal{D},$$

(so that the Green's function G_D for the interior Dirichlet problem exists and is unique) we have

$$\frac{\partial}{\partial N_0} V(x) = -\frac{\partial}{\partial N_0} \int_\Gamma v(s) \frac{\partial G_D(x,s)}{\partial N}\, dS\ , \quad x \in \mathcal{D},$$

where N_0 is a direction which conicides with the normal to Γ on Γ. However, when we let x approach Γ, we do not obtain a bounded operator on H_0 in the present case. In fact, the estimate (A2.15) does not imply

$$\int_\Gamma \left| \frac{\partial u}{\partial N} - \sum_{j=1}^m c_j^{(m)} \frac{\partial \phi_j}{\partial N} \right|^2 dS \leq \delta(\varepsilon)\ , \qquad \delta(\varepsilon) \underset{\varepsilon\to 0}{\to} 0\ , \tag{A2.17}$$

even in the case $u|_\Gamma = f \in C^\infty$. Here u is the solution to the problem

$$(\nabla^2+k^2)u = 0 \ \text{ in } \mathcal{D}\ ,\quad u|_\Gamma = f, \tag{A2.18}$$

and ϕ_j solve the equation

$$(\nabla^2+k^2)\phi_j = 0 \ \text{ in } \mathcal{D}. \tag{A2.19}$$

<u>Proof</u>: If we consider the function $f_\varepsilon \in L^2(\Gamma)$, $\|f_\varepsilon\| < \varepsilon$, then $\|h_\varepsilon - \sum_{j=1}^m c_j \phi_j\| < 2\varepsilon$ where $h_\varepsilon = f + f_\varepsilon$. Let u_ε denote the solution of (1)-(3) with f substituted by h_ε. One can see that $\left\|\frac{\partial u_\varepsilon}{\partial N} - \sum_{j=1}^m c_j \frac{\partial \phi_j}{\partial N}\right\|$

can be as large as one wishes if f_ε is chosen appropriately. In fact, $\frac{\partial u_\varepsilon}{\partial N}$ can be even not defined on Γ. To see this one can take $k=0$ and $\mathcal{D}$ to be a circle of radius 1. Then

$$u_\varepsilon = \sum_{n=-\infty}^{\infty} h_{\varepsilon n} r^{|n|} e^{in\phi} \quad , \quad h_{\varepsilon n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi} h_\varepsilon(\phi)\, d\phi \quad , \quad r<1$$

$$\frac{\partial u_\varepsilon}{\partial N} = \frac{\partial u_\varepsilon}{\partial r} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n| h_{\varepsilon n} r^{|n|-1} e^{in\phi} \tag{A2.20}$$

If $\sum\limits_{n=-\infty}^{\infty} |nh_{\varepsilon n}| = \infty$ the function (A2.20) has no limit in $L^2(\Gamma)$ as $r \to 1-0$. If $\sum\limits_{h=-\infty}^{\infty} |nh_{\varepsilon n}|^2 = c^2 < \infty$, then the limit does exist and $\left\|\frac{\partial u_\varepsilon}{\partial N}\right\| = 2\pi c$ can be as large as one wants if f_ε is chosen appropriately.

Appendix 3. About projection methods.

1. Convergence of projection methods.

Let A be a linear bounded and boundedly invertible operator from a Hilbert space H onto a Hilbert space G. Let P_m be the orthoprojection onto L_m, where L_m is an m-dimensional subspace of H, $L_{m+1} \supset L_m$, and the sequence of the subspaces L_m is limit dense in H, i.e. for any $h \in H$ the distance from h to L_m goes to zero as $m \to \infty$. Let Q_m be the orthoprojection onto M_m, where M_m is an m-dimensional supspace of G, $M_{m+1} \supset M_m$, and the sequence M_m is limit dense in G.

Consider the equation

$$Ah = f \tag{A3.1}$$

and the projection method of its approximate solution

$$Q_m A P_m h_m = Q_m f. \tag{A3.2}$$

The question of when the following statement is true is then of interest:

Equation (A3.2) is uniquely solvable for all sufficiently large m and $\|h_m - h\| \to 0$, $m \to \infty$. (A3.3)

Here h_m is the solution of (A3.2). In the problem described in the Introduction, $h_m = \sum_{j=1}^{m} c_j^{(m)} \phi_j$, and P_m is the orthoprojection in $H = L^2(\Gamma)$ onto the linear span of $\phi_1, \ldots \phi_m$, $G = \ell^2$, Q_m is the orthoprojection in ℓ^2 onto the linear span of the first m coordinate vectors in ℓ^2, i.e. onto the subspace of the vectors whose components f_n vanish for $n>m$. The following theorem, which is a particular case of a more general result from [18] answers the above question.

Theorem 1. (A3.3) holds iff

$$\|Q_m A P_m h\| \geq c\|P_m h\| \quad , \forall m > m_0, \forall h \in H, c > 0, \tag{A3.4}$$

and

$$Q_m A P_m H = Q_m G \quad , \forall m > m_0. \tag{A3.5}$$

Remark 1. If P_m and Q_m are projections onto m-dimensional spaces, where $m=1,2,\ldots$ (this is the case we are interested in in this paper) then (A3.4) implies (A3.5), because the operator $Q_m A P_m$: $P_m H \to Q_m G$ is an injective mapping between two m-dimensional spaces and therefore this mapping is surjective.

Proof: 1) (A3.3) $\Rightarrow$ (A3.4-5). If (A3.3) holds then (A3.2) is uniquely solvable for $m>m_0$ and therefore (A3.5) holds. Furthermore, $(Q_m A P_m)^{-1} Q_m f \to A^{-1} f = h, \forall f \in G$. Therefore $\|(Q_m A P_m)^{-1} Q_m\| \leq c < \infty$. Here and below c denotes various positive constants. Thus $\|P_m h\| = \|(Q_m A P_m)^{-1} Q_m Q_m A P_m h\| \leq c \|Q_m A P_m h\|$, i.e. (A3.4) holds. Note that $(Q_m A P_m)^{-1} \cdot Q_m A P_m = I_m$ where I is the identity in $P_m H$ (not in all of H). 2) (A3.4-5) $\Rightarrow$ (A3.3). From (A3.5) it follows that (A3.2) is uniquely solvable $m>m_0$. To show that $\|h_m - h\| \to 0$, $m \to \infty$, consider the equalities

$$Q_m A[P_m h + (I - P_m)h] = Q_m f$$

$$Q_m A P_m h_m = Q_m f \, ,$$

which imply that

$$Q_m A P_m (h_m - P_m h) = Q_m A(I - P_m)h. \tag{A3.6}$$

Since the sequence of the subspaces L_m is limit dense in H one

has $(I-P_m)h\to 0, \forall h\in H$. Therefore (A3.6) and (A3.4) imply that

$$\|h_m - P_m h\| = \|P_m(h_m - P_m h)\| \leq c\|Q_m A(I-P_m)h\| \to 0, \quad m\to\infty. \tag{A3.7}$$

Thus

$$\|h-h_m\| \leq \|h-P_m h\| + \|P_m h - h_m\| \to 0, \quad m\to 0. \tag{A3.8}$$

This completes the proof, which is borrowed from [18] (see also [22]).

2. Stability of the projection methods.

Suppose that (A3.3) holds for the operator A in (A3.1).

1) Will it hold for A+B where $\|B\|<\delta$ and $\delta>0$ is sufficiently small? The answer is yes.

2) Will it hold for A+B where B is compact and A+B is boundedly invertible? The answer is yes.

The proofs can be found in [18]. Since they are simple we give them here for convenience of the reader. 1) Let $\delta=c-\varepsilon$, $\|B\|\leq\delta$, where c is the constant in (A3.4), $0<\varepsilon<c$. Then $\|Q_m(A+B)P_m h\| \geq$ $\geq c\|P_m h\| - \delta\|P_m h\| = \varepsilon\|P_m h\|$. For the case when Q_m and P_m are finite dimensional projections onto m-dimensional spaces, Theorem 1 is applicable. (See Remark 1.) In the general case it is not difficult to show that (A3.5) holds, i.e. that the operator $Q_m(A+B)P_m$: $P_m H\to Q_m G$ is invertible:

$$Q_m(A+B)P_m = Q_m A P_m\left[I_m + (Q_m A P_m)^{-1} Q_m B P_m\right],$$

and $\|(Q_m AP_m)^{-1}Q_m BP_m\| \leq \frac{c-\varepsilon}{c} <1$. Therefore conditions (A3.4-5) are satisfied by the operator A+B and (A3.3) holds for the operator A+B. 2) If B is compact then $\|(Q_m AP_m)^{-1}Q_m B - A^{-1}B\| \to 0$, $m\to\infty$. because $(Q_m AP_m)^{-1}Q_m \to A^{-1}$ strongly. If A+B is invertible then so is $I+A^{-1}B$, and $\|P_m h + A^{-1}BP_m h\| > c_1\|P_m h\|$. Therefore

$$\|Q_m(A+B)P_m h\| = \|Q_m A P_m[P_m h + (Q_m A P_m)^{-1} Q_m B P_m h]\|$$

$$\geqslant c\|P_m h + (Q_m A P_m)^{-1} Q_m B P_m h\| \geqslant c\|P_m h + A^{-1} B P_m h\|$$

$$- c\|[(Q_m A P_m)^{-1} Q_m B - A^{-1}B] P_m h\|$$

$$\geqslant \frac{cc_1}{2} \|P_m h\| \quad , \ \forall m > m_0 .$$

Thus, condition (A3.4) holds for A+B. To check condition (A3.5) one notes that $Q_m A P_m$ is invertible, $Q_m B P_m$ is compact and $Q_m A P_m +$ $+ Q_m B P_m$ is one to one by virtue of (A3.4). By Fredholm's alternative one concludes that $Q_m (A+B) P_m$ is invertible and (A3.5) holds.

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Table 1

Eccentricity	Truncation size 5		10		20		40	
q=0.1	6	(4)	60	(40)	$8\cdot10^3$	$(5\cdot10^3)$	$2\cdot10^8$	$(1\cdot10^8)$
q=1/3	$1\cdot10^2$	(80)	$7\cdot10^4$	$(4\cdot10^4)$	$3\cdot10^{10}$	$(2\cdot10^{10})$	$>10^{16}$	$(>10^{16})$
q=1	$8\cdot10^3$	$(5\cdot10^3)$	$1\cdot10^9$	$(6\cdot10^8)$	$>10^{16}$	$(>10^{16})$	$>10^{16}$	$(>10^{16})$

The condition number κ as a function different truncation sizes and eccentricities for the model problem (see (41)). The corresponding condition number κ for the normalized functions $\phi_m\cdot(\phi_m,\phi_m)^{-\frac{1}{2}}$ is given in parentheses.

Table 2

Truncation size (ℓ_{max}; even or odd ℓ-values)

	5 even	5 odd	9 even	9 odd	19 even	19 odd
ka=4 kb=2						
ψ_n	30(3)	400(5)	$2\cdot10^7$(30)	$1\cdot10^9$(50)	$>10^{16}(3\cdot10^4)$	$>10^{16}(6\cdot10^4)$
Reψ_n	30(20)	100(10)	$5\cdot10^6$(200)	$2\cdot10^8$(400)	$>10^{16}(1\cdot10^6)$	$>10^{16}(3\cdot10^6)$
χ_n	$1\cdot10^3$(5)	$1\cdot10^3$(8)	$2\cdot10^6$(50)	$2\cdot10^6$(100)	$1\cdot10^{14}(4\cdot10^4)$	$1\cdot10^{14}(1\cdot10^5)$
Y_n	3(3)	3(3)	4(4)	4(4)	4(4)	4(4)
ka=6 kb=2						
ψ_n	30(3)	40(5)	$2\cdot10^7$(40)	$1\cdot10^9$(90)	$>10^{16}(1\cdot10^5)$	$>10^{16}(3\cdot10^5)$
Reψ_n	400(400)	20(80)	$4\cdot10^4$(500)	$8\cdot10^5(1\cdot10^3)$	$>10^{16}(1\cdot10^8)$	$>10^{16}(3\cdot10^8)$
χ_n	$2\cdot10^4$(6)	$2\cdot10^4$(10)	$3\cdot10^6$(100)	$3\cdot10^6$(200)	$5\cdot10^{14}(2\cdot10^5)$	$4\cdot10^{14}(4\cdot10^5)$
Y_n	7(7)	6(6)	9(9)	9(9)	10(10)	10(10)
ka=10 kb=2						
ψ_n	30(3)	300(5)	$2\cdot10^7$(50)	$1\cdot10^9$(100)	$>10^{16}(2\cdot10^5)$	$>10^{16}(4\cdot10^5)$
Reψ_n	30(30)	3(2)	60(30)	600(200)	$5\cdot10^{14}(8\cdot10^8)$	$7\cdot10^{15}(2\cdot10^9)$
χ_n	$3\cdot10^3$(6)	$2\cdot10^3$(10)	$5\cdot10^6$(100)	$4\cdot10^6$(300)	$1\cdot10^{14}(3\cdot10^5)$	$7\cdot10^{14}(8\cdot10^5)$
Y_n	20(20)	10(10)	30(30)	30(30)	40(40)	40(40)

The condition number κ as a function of truncation sizes and eccentricities for three different spherical waves $\{\psi_n\}$, $\{\mathrm{Re}\psi_n\}$, $\{\chi_n\}$ and the spherical harmonics $\{Y_n\}$. The corresponding condition number for the normalized functions is given in parentheses.
$m=0$ in all cases.

APPENDIX VI

Variational principles for resonances. II

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(Received 23 June 1981; accepted for publication 7 August 1981)

Variational principles for calculating the complex poles of Green's function are given. Convergence of the numerical procedure is proved.

PACS numbers: 03.65.Nk

I. INTRODUCTION

This note is a continuation of Ref. 1, where the following problem was considered:

$$(-\nabla^2 - k^2)u = 0 \quad \text{in} \quad \Omega, \ u|_\Gamma = 0. \tag{1}$$

Here Ω is an exterior domain, Γ is its closed smooth boundary, and $D = \mathbb{R}^3 \backslash \Omega$ is bounded. Problem (1) has nontrivial solutions if and only if ($\equiv$iff) k is a complex pole k_q of the Green's function $G(x,y,k)$ of the exterior Dirichlet problem. In Ref. 1 a stationary variational principle for resonances, i.e., complex poles k_q, was given

$$k^2 = \mathrm{st}\{\langle \nabla u, \nabla u\rangle / \langle u,u\rangle\}, \tag{2}$$

where st is the symbol of stationary value,

$$\langle u,v\rangle = \lim_{\epsilon \to +0} \int \exp(-\epsilon r \ln r) u(x) v(x)\, dx,$$

$$\int = \int_\Omega, \quad r = |x|. \tag{3}$$

In Ref. 1 the test functions for (2) were taken in the form

$$u_N = r^{-1}\exp(ikr) \sum_{j=0}^{N} \sum_{m=-j}^{j} r^{-j} Y_{jm}(n) c_{jm} g(x), \tag{4}$$

where $n = x|x|^{-1}$, Y_{jm} are the spherical harmonics, c_{jm} are constants, k is a parameter, and $g(x) \geq 0$ is a fixed smooth function vanishing on Γ and equal to 1 outside of some ball containing D. It was not proved in Ref. 1 that the numerical procedure suggested there converges. The question formulated in Ref. 1 concerning the justification of the numerical approach is still open. The purpose of this note is to formulate another variational principle for calculating the complex poles k_q and to prove the convergence of the numerical procedure. The method in Ref. 1 is similar to Ritz's method. The method suggested in this note is similar to Trefftz's method. The advantage of this method is that one deals with the compact operators, while in Ref. 1 the operator was not compact. Our construction is natural in the framework of the singularity and eigenmode expansion methods.[2] The convergence of the method will be proved. A result which is of general interest, as it seems to the author, is a construction of a stationary variational principle and a proof of convergence for a class of non-self-adjoint symmetric operators ($B^* = \bar{B}$), which occur frequently in the scattering theory.

II. A VARIATIONAL PRINCIPLE

The starting point is the following observation: k is a complex pole of $G(x,y,k)$ iff the equation

$$Af \equiv \int_\Gamma g(s,t,k) f(t)\, dt = 0, \ \mathrm{Im} k < 0,$$

$$g(s,t,k) = \exp(ik|s-t|)/(4\pi|s-t|), \tag{5}$$

has a nontrivial solution. This observation and some consequences are discussed in Ref. 3. For the convenience of the reader let us note that

$$G = g - \int_\Gamma g(x,t,z) \frac{\partial G(t,y,z)}{\partial N_t}\, dt, \tag{6}$$

where N_t is the unit outer normal to Γ at the point t. If k is a complex pole of G of order r one can multiply (6) by $(z-k)^r$ and take $z \to k$ and $x = s \in \Gamma$. This yields Eq. (5) (see Ref. 3, pp. 290–291) with $f \not\equiv 0$.

Let us formulate the following variational principle

$$F(f) \equiv |Af|_1^2 = \min, \ \|f\| = 1, \tag{7}$$

where $|f|_p$ is the norm in the Sobolev space $H_p = W_2^p(\Gamma)$, $\|f\| = |f|_0$. From the above observation it follows that (7) has solutions and the min is zero if $k = k_q$, where k_q are the poles of $G(x,y,k)$. If $k \neq k_q$ then $\inf_{\|f\|=1} |Af|_1 > 0$. Indeed, if there exists a sequence $\|f_n\| = 1$, $|Af_n|_1 \to 0$, then $f_n \to f$, $\|f\| = 1$, $Af = 0$, and therefore $k = k_q$ (see Ref. 3, p. 291). The only point which is to be explained is the convergence in H: $f_n \to f$. In Ref. 3 it is explained that A is a pseudodifferential operator of order -1, that is,

$$a_1|f|_{p-1} \leq |Af|_p \leq a_2|f|_{p-1}. \tag{8}$$

Here $a_1, a_2 > 0$ are some constants, $-\infty < p < \infty$ if $\Gamma \subset C^\infty$, and the fact that $k \neq k_q$ was used essentially: if $k \neq k_q$ then $\ker A \equiv \{f: Af = 0\} = \{0\}$ and A maps H_p onto H_{p+1}. If $|Af_n|_1 \to 0$ and $\|f_n\| = 1$, then (8) with $p = 1$ shows that $\|f_n\| \to 0$. This contradicts the equation $\|f_n\| = 1$. Therefore

$$\inf_{\|f\|=1} |Af|_1 > 0 \quad \text{if} \quad k \neq k_q. \tag{9}$$

Consider a numerical method for solving problem (7). Let $\{f_j\}$ be a basis of H,

$$f = f^{(n)} \equiv \sum_{j=1}^{n} c_j f_j. \tag{10}$$

The necessary condition for $F(f)$ to be minimal and $\min F(f^{(n)}) = 0$, $\|f^{(n)}\| = 1$, yields:

$$\sum_{m=1}^{n} a_{jm} c_m = 0 \quad 1 \leq j \leq n, \tag{11}$$

[a)]Supported by AFOSR 80024. AMS subject classification 47A10, 78A45, 81F05, 35J05.

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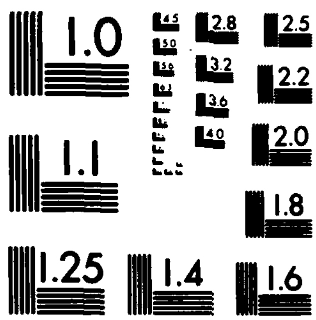

MICROCOPY RESOLUTION TEST CHART
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$$a_{jm} = a_{jm}(k) = (Af_m, Af_j)_1 \quad \sum_{j=1}^{n} |c_j|^2 > 0. \tag{12}$$

Thus

$$\det a_{jm}(k) = 0 \quad 1 < j,m < n. \tag{13}$$

Let $k_q^{(n)}$ denote the roots of Eq. (13). Our first result is

Theorem 1: There exists $\lim_{n\to\infty} k_q^{(n)} = k_q$, and k_q are the poles of Green's function $G(x,y,k)$. Every pole k_q is a limit of a sequence $k_q^{(n)}$, where $k_q^{(n)}$ are the roots of (13). Convergence is uniform in q for any finite interval $1<q<Q$.

Proof: We will prove that: (i) Eq. (13) has roots in the circle $|k-k_q|<\epsilon$ for any fixed $\epsilon>0$ however small if $n>n(\epsilon)$ is large enough. (ii) If $n>n(\epsilon)$ and there are no points k_q in the circle $|k-z|<\epsilon$ then Eq. (13) has no roots in the circle $|k-z|<\epsilon$. An important part of the proof is the reduction of the problem to the problem with the operator $I+T(k)$, where $T(k)$ is compact.

Let us fix $\epsilon>0$ such that in the circle $\{k:|k-k_q|<\epsilon\} = K_\epsilon$ there are no other poles. The operator $A=A(k)$ can be written as

$$A(k) = A_0[I+T(k)], \tag{14}$$

where

$$A_0 = A(0), \quad A_0^* = A_0 > 0 \quad \text{in} \quad H = L^2(\Gamma),$$

$$A_0 f = \int_\Gamma \frac{f\,dt}{4\pi r_{st}}, \tag{15}$$

and

$$T(k) = A_0^{-1}[A(k) - A_0]. \tag{16}$$

The operator A_0 is a bijection of H_p onto H_{p+1};

$$b_1|f|_0 < |A_0 f|_1 < b_2|f|_0, \; b_1, b_2 = \text{const} > 0, \tag{17}$$

while $T(k)$ is compact as a map $H_p \to H_p$ (see Ref. 3 for details) because $A(k)-A_0$ is an operator with a nonsingular kernel. Let us rewrite functional (7) as

$$F(f) = |A_0(I+T)f|_1^2 = \min, \quad \|f\| = 1. \tag{18}$$

From (18) and (17) it follows that the problem (7) is equivalent to

$$F_0(f) = |(I+T)f|_0^2 = \min, \quad \|f\| = 1. \tag{19}$$

The matrix of the system (11) can be written as

$$a_{jm} = ((I+T)f_m, (I+T)f_j), \tag{20}$$

where $(.,.)$ denotes the scalar product which is metrically equivalent to the scalar product in H. This means that $d_1(f,f)_0 < (f,f) < d_2(f,f)_0$, where $d_1>0$ and d_2 are constants, $f\in H$ is arbitrary. In the sequel we will not discriminate between $(.,.)$ and $(.,.)_0$. This is possible because $((I+T)f, (I+T)f)$ and $((I+T)f, (I+T)f)_0$ attain their zero values simultaneously. The system (11) can be considered as the system which corresponds to the Ritz method for functional (19) with the test functions $\{f_j\}$. This completes the reduction of the original problem to the problem with the operator $I+T(k)$, where $T(k)$ is a compact analytic-in-k operator function on H. To prove (i) let us assume that for a fixed $\epsilon>0$ and k_q and all n there are no roots $k_q^{(n)}$ of Eq. (13) in the circle $|k-k_q|<\epsilon$. The system (11) with the matrix (20) says that

$$((I+T)f^{(n)}, (I+T)f_j) = 0 \quad 1<j<n, \; f^{(n)} \neq 0, \tag{21}$$

where

$$f^{(n)} = \sum_{j=1}^{n} c_j f_j. \tag{22}$$

In particular, our assumption means that $(\tilde{T} \equiv T + T^* + T^*T, \; I+\tilde{T} = (I+T^*)(I+T))$,

$$(I + P_n\tilde{T}(k))f^{(n)} = 0 \Rightarrow f^{(n)} \equiv 0, \quad |k-k_q|<\epsilon, \tag{23}$$

where P_n denotes the projection in H onto the linear span of $\{f_1,...,f_n\}$. Equation (23) says that $I+P_n\tilde{T}(k)$ is invertible in the circle $|k-k_q|<\epsilon$. If n is large enough this implies that $I+\tilde{T}(k)$ is invertible in the circle $|k-k_q|<\epsilon$, because (*) $\|I+\tilde{T}(k) - (I+P_n\tilde{T}(k))\| \to 0$ as $n\to\infty$. This is a contradiction since $I+\tilde{T}(k_q)$ is not invertible. Let us explain (*). We need to show that $\|(I-P_n)\tilde{T}\| \to 0$ as $n\to\infty$. Since $\tilde{T}(k)$ is compact it can be written as $T_N + B_N$, where $\|B_N\| < d_N$, $d_N \to 0$ as $N\to\infty$, and T_N is a finite-dimensional operator. It is sufficient to prove that $\|(I-P_n)T_N\| \to 0$ as $n\to\infty$. Without loss of generality one can assume that T_N is a one-dimensional operator, $T_N f = (f,v)u$. Then

$$\|(I-P_n)T_N f\| = \|(I-P_n)u\| \, |(f,v)|$$
$$< \|f\| \, \|v\| \, \|u - P_n u\| \to 0 \quad \text{as} \quad n\to\infty, \tag{24}$$

since $P_n \to I$ strongly. Thus the statement (i) is proved. Note that the orthogonality of P_n is not used in (24). In order to prove (ii) we suppose that for any $\epsilon_n>0$, $\epsilon_n\to 0$, Eq. (13) has a root $k^{(n)}$ in the circle $|k-z|<\epsilon_n$ and show that under this assumption z has to be a pole of the Green's function. The assumption means that

$$[I+T(k^{(n)})]f^{(n)} = 0, \quad \|f^{(n)}\| = 1, \quad k^{(n)} \to z. \tag{25}$$

Since $\|f^{(n)}\| = 1$, one can extract a weakly convergent in H subsequence which is denoted again $f^{(n)}$: $f^{(n)} \rightharpoonup f$ ($\rightharpoonup$ means weak convergence). Since $T(k)$ is compact the sequence $T(z)f^{(n)}$ converges strongly in H:

$$T(z)f^{(n)} \to T(k)f. \tag{26}$$

On the other hand,

$$\|T(k_n) - T(z)\| \to 0. \tag{27}$$

From (25)–(27) it follows that

$$f^{(n)} \to f, \quad \|f\| = 1, \tag{28}$$

and

$$[I+T(z)]f = 0, \quad \|f\| = 1. \tag{29}$$

The proof is complete.

III. DISCUSSION

The variational principles (19) and (18) can be viewed as the least square method. Let us consider instead of (13) and (20) the following equation:

$$\det b_{jm}(k) = 0, 1<j,m<n, \quad b_{jm} \equiv ((I+T(k))f_m, f_j). \tag{30}$$

Arguments similar to the ones given in Ref. 3, pp. 192–193 show that: (i) For any $\epsilon>0$ and k_q there exists a root $\tilde{k}_q^{(n)}$ of Eq. (30). such that $|k_q - \tilde{k}_q^{(n)}| < \epsilon$ if $n>n(\epsilon)$. (ii) If $\tilde{k}_q^{(n)}$ is a sequence of the roots of Eq. (30) and $\tilde{k}_q^{(n)} \to k_q$ as $n\to\infty$, then k_q is a pole of the Green's function. Equation (30) can be viewed as a necessary condition for the linear system of the

Galerkin method for the equation $(I + T(k))f = 0$ to have a nontrivial solution. The Galerkin equation is of the form

$$(f^{(n)} + T(k)f^{(n)}, f_j) = 0 \quad 1 < j < n, \tag{31}$$

where $f^{(n)}$ is defined in (22). The basic idea is that the poles k_q are the points at which the operator $I + T(k)$ is not invertible. These points can be found by the Galerkin method, by minimizing functional (19) or by some other method. It is interesting to note that the Galerkin equation (31) can be obtained also as a necessary condition for the stationary variational principle

$$((I + T(k))f, f) = \mathrm{st}, \quad \|f\| > 0, \tag{32}$$

where st means stationary value. This is not true for an arbitrary operator, but the operator $B = I + T(k)$ is a symmetric non-self-adjoint operator on $H = L^2(\Gamma)$, that is

$$\overline{B^*} = B \quad \text{or} \quad B(s,t) = B(t,s)\left[\neq \overline{B(t,s)}\right]. \tag{33}$$

Therefore the necessary condition for (32), which can be written as

$$(Bf, h) + (Bh, f) = 0 \quad \text{for all} \quad h \in H, \tag{34}$$

yields

$$0 = (Bf, h) + (\overline{B^* f}, \bar{h}) = (Bf, h) + (B\bar{f}, \bar{h}). \tag{35}$$

Let $h = v$, where $v \in H$ is an arbitrary real-valued function. Then (35) says that

$$0 = B(f + \bar{f}). \tag{36}$$

Let $h = iv$. Then (35) says that

$$0 = B(f - \bar{f}). \tag{37}$$

From (36) and (37) it follows that the equation

$$Bf = (I + T(k))f = 0, \quad \|f\| > 0 \tag{38}$$

is a necessary condition for (32).

Our aim is to show that Eq. (31) is a necessary condition for the problem

$$(Bf, f) = \mathrm{st}, \quad \|f\| > 0. \tag{39}$$

Let us take $f = f^{(n)}$ and rewrite (39) as

$$\sum_{j,m=1}^{n} b_{jm} c_m \bar{c}_j = \mathrm{st}, \quad b_{jm} = (Bf_m, f_j). \tag{40}$$

In general assumption (33) does not imply the equality $b_{jm} = b_{mj}$. Therefore the following lemma is of use.

Lemma 1: Assume (33) and

$$f_j = \bar{f}_j, \quad j = 1,2,\cdots. \tag{41}$$

Then

$$b_{jm} = b_{mj}. \tag{42}$$

The proof is immediate.

Proposition 1: Assume (33) and (41). Then a necessary condition for (40) is the system (31).

Proof: The operator $B = I + T(k)$ satisfies (33). From this and Lemma 1, Proposition 1 follows.

Remark 1: The results of Sec. III give a convergent numerical scheme for a stationary variational principle (32) with a compact operator T satisfying condition (33), i.e., symmetric non-self-adjointness. Such operators occur frequently in the scattering theory. A simple example is problem (1). There are other examples in Ref. 4.

Remark 2: A numerical scheme for calculating the resonances based on theorem 1 is as follows: (1) Calculate matrix a_{jm} by formula (11). (2) Find roots of Eq. (13). The corresponding solutions of (1) can also be calculated by this numerical procedure: Find $f^{(n)}$ by formula (10) and $u^{(n)} = Af^{(n)}$ is the approximate solution of (1), which converges to the exact solution of (1) as $n \to \infty$. This exact solution is of the form $u = A(k_q)f$, $f = \lim f^{(n)}$ as $n \to \infty$ and lim here means the limit in $H = L^2(\Gamma)$.

Remark 3: For numerical calculations instead of principle (32) one should use the equivalent principle

$$(A(k)f, f) = \mathrm{st}, \quad \|f\| > 0. \tag{43}$$

The equivalence of (43) and (32) follows from the fact that the necessary condition for (43) is the equation

$$A(k)f = A_0(I + T(k))f = 0, \quad \|f\| > 0, \tag{44}$$

which is equivalent to the necessary condition (38) for (32) because $\ker A_0 = \{0\}$. If one takes $f = f^{(n)}$ as in (22), then the analog of (30) is

$$\det(A(k)f_m, f_j) = 0 \quad 1 < m, j < n, \tag{45}$$

and the convergence of the numerical procedure follows from the arguments given for Eq. (30).

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